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# Duality Transformations in Supersymmetric Yang–Mills Theories coupled to Supergravity

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## Abstract

We consider duality transformations in  $N = 2$ ,  $d = 4$  Yang–Mills theory coupled to  $N = 2$  supergravity. A symplectic and coordinate covariant framework is established, which allows one to discuss stringy ‘classical and quantum duality symmetries’ (monodromies), incorporating  $T$  and  $S$  dualities. In particular, we shall be able to study theories (like  $N = 2$  heterotic strings) which are formulated in symplectic basis where a ‘holomorphic prepotential’  $F$  does not exist, and yet give general expressions for all relevant physical quantities. Duality transformations and symmetries for the  $N = 1$  matter coupled Yang–Mills supergravity system are also exhibited. The implications of duality symmetry on all  $N > 2$  extended supergravities are briefly mentioned. We finally give the general form of the central charge and the  $N = 2$  semiclassical spectrum of the dyonic BPS saturated states (as it comes by truncation of the  $N = 4$  spectrum).

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# 1 Introduction

Recently, proposals for the quantum moduli space of  $N = 2$  rigid Yang–Mills theories [1] have been given in terms of particular classes of genus  $r$  Riemann surfaces parametrized by  $r$  complex moduli[2],  $r$  being the rank for the gauge group  $G$  broken to  $U(1)^r$  for generic values of the moduli. The effective action for such theories, with terms up to two derivatives, is described by  $N = 2$  supersymmetric lagrangians of  $r$  abelian massless vector multiplets[3], whose dynamics is encoded in a holomorphic prepotential  $F(X^A)$ , function of the moduli coordinates  $X^A$  ( $A = 1, \dots, r$ ). According to Seiberg and Witten [1] this effective theory has classical, perturbative and non perturbative duality symmetries which reflect on monodromy properties of certain holomorphic symplectic vectors  $(X^A, F_A(X))$ , eventually related to periods of holomorphic one-forms[1]

$$\omega = X^A \alpha_A + F_A \beta^A, \quad (1.1)$$

where  $\alpha_A, \beta^A$  is a basis for the  $2r$  homology cycles of a genus  $r$  Riemann surface. The Picard–Fuchs equations satisfied by the holomorphic vector one-form  $U_i = (\partial_i X^A, \partial_i F_A)$  ( $i = 1, \dots, r$ ) can be regarded as differential identities for “rigid special geometry” [4]. To attach a particular algebraic curve to “rigid special geometry” is therefore equivalent to exactly compute the holomorphic data  $U_i$ , and thus to exactly reconstruct the effective action for the self interaction of the  $r$  massless gauge multiplets once the massive states, both perturbative and non perturbative, have been integrated out. Indeed it is a virtue of  $N = 2$  supersymmetry that all the couplings in the effective Lagrangian, including 4-fermion terms, can be computed purely in terms of the holomorphic data. Quite remarkably the quantum monodromies dictate the monopole and dyon spectrum of the effective theory [1,2] which turns out to be “dual” to non-perturbative instanton effects [5] in the original  $G$ -invariant microscopic theory [6,7].

This paper considers several issues in order to extend the approach pursued in the rigid case to the more challenging case of coupling an  $N = 2$  Yang–Mills theory to gravity. In particular we shall include in the  $N = 2$  supergravity theory a dilaton–axion vector multiplet which is an essential ingredient to describe effective  $N = 2$  theories which come from the low energy limit of  $N = 2$  heterotic string theories in four dimensions [8]. Another ingredient is the extension of the “classical monodromies” to  $N = 2$  local supersymmetry. For rigid theories the classical metric is essentially the Cartan matrix of the group  $G$  and the classical monodromies are

related to the Weyl group of the Cartan subalgebra of  $G$  [2]. For  $N = 2$  supergravity theories coming from  $N = 2$  heterotic strings, the classical metric of the moduli space of the pure gauge sector is based on the homogeneous space  $O(2, r)/O(2) \times O(r)$  [3, 8–10] and the classical monodromies are related to the  $T$ -duality group  $O(2, r; \mathbb{Z})$  which in particular is an invariance of the massive charged states [11]. This state of affair is quite analogous to the analysis performed by Sen and Schwarz [12] for the  $N = 4$  heterotic string compactifications, in which case an exact quantum duality symmetry  $SL(2, \mathbb{Z}) \times O(6, r; \mathbb{Z})$  was conjectured [12–16] and a resulting spectrum for BPS states with both electric and magnetic states was proposed. In the  $N = 4$  theory the  $SL(2, \mathbb{Z}) \times O(6, r; \mathbb{Z})$  symmetry, using general arguments [17, 18], has a natural embedding in  $Sp(2(6 + r); \mathbb{Z})$ , acting on the  $6 + r$  vector self-dual field strengths  $\mathcal{F}_{\mu\nu}^{+A}$  and their “dual” defined through  $G_{+A}^{\mu\nu} \equiv -i \frac{\delta \mathcal{L}}{\delta \mathcal{F}_{\mu\nu}^{+A}}$ . In generic  $N = 2$  theories, because of quantum corrections [19, 20], we do not expect such factorized  $S - T$  duality to occur anymore [4]. Indeed this can be argued with a pure supersymmetry argument, related to the fact that once the classical moduli space  $O(2, r)/O(2) \times O(r)$  is deformed by quantum corrections, then the factorized structure with the dilaton degrees of freedom is lost and a non trivial moduli space, mixing the  $S$  and  $T$  degrees of freedom should emerge. This result is in fact a consequence of a theorem on “special geometry” [21, 22] which asserts that the only factorized special manifolds are the  $\frac{SU(1,1)}{U(1)} \times \frac{O(2,r)}{O(2) \times O(r)}$  series, which precisely describe the “classical moduli space” of  $S - T$  moduli. Because of the coupling to gravity, the symplectic structure and identification of periods, coming from special geometry, is also remarkably different from rigid special geometry. Indeed the interpretation of  $(X^\Lambda, F_\Lambda)$ ,  $\Lambda = 0, 1, \dots, r + 1$  as periods of algebraic curves is no longer appropriate to genus  $r$  Riemann surfaces, as it can be seen from the Picard–Fuchs equations [23, 24] and from the form of the metric  $g_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log i(\bar{F}_A X^A - \bar{X}^A F_A)$  of the moduli space [23–29]. In fact special geometry is known to be appropriate to a particular class of complex manifolds (Calabi–Yau manifolds or their mirrors) and to describe the deformations of the complex structure [23]. It is therefore tempting to argue that the quantum moduli space including  $S - T$  duality and its monodromies is related to 3-manifolds (or their mirrors) with  $h_{(2,1)} = r + 1$ .

The paper is organized as follows: In chapter 2 we give a résumé of rigid theories, also discussing duality for the fermionic sector and the physical significance of monodromies and geometrical data, such as the holomorphic tensor  $C_{ijk}$ , related to the gaugino anomalous magnetic moment. In chapter 3 we describe in detail the coupling to gravity, the extension of duality to the fermionic sector and the existence of

symplectic bases which do not admit a prepotential function  $F$ , as it occurs in certain formulations of  $N = 2$  supergravities coming from  $N = 2$  heterotic strings. The general form of duality transformations and symmetries as they occur in  $N = 1$  locally supersymmetric Yang–Mills theories coupled to matter is also described. In chapter 4 we use such a formulation where all the perturbative duality symmetries become invariances of the action. Then, we discuss the implementation of duality symmetries in  $N > 2$  extended supergravities for the spectrum of dyonic states. In chapter 5 we analyze classical and quantum duality symmetries and give generic formulae for the spectrum of the BPS states and the “semiclassical formulae” when the non perturbative spectrum is computed in terms of the “classical periods”. The explicit expression for the  $r = 2$  case is given as an example, and the special occurrence of enhanced symmetry points is described. The paper ends with some concluding remarks.

## 2 Résumé of rigid special geometry

### 2.1 Basics

$N = 2$  supersymmetric gauge theory on a group  $G$  broken to  $U(1)^r$ , with  $r = \text{rank } G$ , corresponds to a particular case of the most general  $N = 1$  coupling of  $r$  chiral multiplets  $(X^A, \chi^A)$  to  $r$   $N = 1$  abelian vector multiplets  $(\mathcal{A}_\mu^A, \lambda^A)$  in which the Kähler potential  $K$  and the holomorphic kinetic term function  $f_{AB}(X^A)$  are given by

$$\begin{aligned} K &= i(\overline{F}_A X^A - F_A \overline{X}^A) \quad , \quad (F_A = \partial_A F) \\ f_{AB} &= \partial_A \partial_B F \equiv F_{AB} \end{aligned} \tag{2.1}$$

in terms of the single prepotential  $F(X)$ [3]. One can show that the Kähler geometry is constrained because the Riemann tensor satisfies the identity [26,4]

$$R_{A\overline{B}C\overline{D}} = -\partial_A \partial_C \partial_P F \partial_{\overline{B}} \partial_{\overline{D}} \partial_{\overline{Q}} \overline{F} g^{P\overline{Q}} \quad , \tag{2.2}$$

with

$$g_{P\overline{Q}} = \partial_P \partial_{\overline{Q}} K = 2 \text{Im } \partial_P \partial_{\overline{Q}} F \quad . \tag{2.3}$$

The lagrangian has the form

$$\begin{aligned} \mathcal{L} &= g_{A\overline{B}} \partial_\mu X^A \partial_\mu \overline{X}^{\overline{B}} + (g_{A\overline{B}} \lambda^{IA} \sigma^\mu \mathcal{D}_\mu \overline{\lambda}_I^{\overline{B}} + \text{h.c.}) \\ &+ \text{Im } (F_{AB} \mathcal{F}_{\mu\nu}^{-A} \mathcal{F}_{\mu\nu}^{-B}) + \mathcal{L}_{\text{Pauli}} + \mathcal{L}_{4\text{-fermi}} \quad , \end{aligned} \tag{2.4}$$

where  $A, B, \dots$  run on the adjoint representation of the gauge group  $G$ ,  $I = 1, 2$  and  $\mathcal{F}_{\mu\nu}^{+A} = \mathcal{F}_{\mu\nu}^A - \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\mathcal{F}^{A\rho\sigma}$  (and  $\mathcal{F}_{\mu\nu}^{-A} = \overline{\mathcal{F}}_{\mu\nu}^{+A}$ ). As we shall see, also  $\mathcal{L}_{\text{Pauli}}$  and  $\mathcal{L}_{4\text{-Fermi}}$  contain the function  $F$  and its derivatives up to the fourth.

The previous formulation, derived from tensor calculus, is incomplete because it is not coordinate covariant. It is written in a particular coordinate system (“special coordinates”) which is not uniquely selected. In fact, eq.(2.1) is left invariant under particular coordinate changes of the  $X^A \rightarrow \tilde{X}^A$  with some new function  $\tilde{F}(\tilde{X})$  described by

$$\begin{aligned}\tilde{X}^A(X) &= A^A_B X^B + B^{AB} F_B(X) + P^A \\ \tilde{F}_A(\tilde{X}^A(X)) &= C_{AB} X^B + D_A^B F_B(X) + Q_A,\end{aligned}\tag{2.5}$$

where  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$  is an  $Sp(2r, \mathbb{R})$  matrix

$$A^T C - C^T A = 0, \quad B^T D - D^T B = 0, \quad A^T D - C^T B = \mathbb{1},\tag{2.6}$$

and  $P^A, Q_A$  can be complex constants which from now on will be set to zero.

It can be shown that a function  $\tilde{F}$  exists such that [3]

$$\tilde{F}_A = \frac{\partial \tilde{F}}{\partial \tilde{X}^A},\tag{2.7}$$

provided the mapping  $X^A \rightarrow \tilde{X}^A$  is invertible.

It is well known that the equations of motion and the Bianchi identities [3][17][18]

$$\begin{aligned}\partial^\mu \text{Im } \mathcal{F}_{\mu\nu}^{-A} &= 0 && \text{Bianchi identities} \\ \partial_\mu \text{Im } G_{-A}^{\mu\nu} &= 0 && \text{Equations of motion}\end{aligned}\tag{2.8}$$

transform covariantly under (2.5) (with  $P^A = Q_A = 0$ ), so that  $(\mathcal{F}_{\mu\nu}^{-A}, G_{-A}^{\mu\nu})$  is a symplectic vector. Here,  $G_{-A}^{\mu\nu} \equiv i \frac{\delta \mathcal{L}}{\delta \mathcal{F}_{\mu\nu}^{-A}} = \overline{\mathcal{N}}_{AB} \mathcal{F}_{\mu\nu}^{-B} + \text{fermionic terms}$ , where we have set  $F_{AB} = \overline{\mathcal{N}}_{AB}$  in order to unify the notations to the gravitational case[3]. The transformations (2.5) leave invariant the whole lagrangian but the vector kinetic term. Indeed, neglecting for the moment fermion terms (see section 2.2) and setting

for simplicity  $\mathcal{F}_{\mu\nu}^{-A} = \mathcal{F}^A$  and  $G_{-A}^{\mu\nu} = G_A$  the vector kinetic lagrangian transforms as follows

$$\begin{aligned} \text{Im } \mathcal{F}^A \overline{\mathcal{N}}_{AB} \mathcal{F}^B &\rightarrow \text{Im } \tilde{\mathcal{F}}^A \tilde{G}_A = \\ &= \text{Im } (\mathcal{F}^A G_A + 2\mathcal{F}^A (C^T B)_A^B G_B + \\ &+ \mathcal{F}^A (C^T A)_{AB} \mathcal{F}^B + G_A (D^T B)^{AB} G_B) . \end{aligned} \quad (2.9)$$

If  $C = B = 0$  the lagrangian is invariant. If  $C \neq 0, B = 0$  it is invariant up to a four-divergence. In presence of a topologically non-trivial  $\mathcal{F}_{\mu\nu}^{-A}$  background,  $(C^T A)_{AB} \int \text{Im } \mathcal{F}_{\mu\nu}^{-A} \mathcal{F}_{\mu\nu}^{-B} \neq 0$ , one sees that in the quantum theory duality transformations must be integral valued in  $Sp(2r, \mathbb{Z})[1]$  and transformations with  $B = 0$  will be called perturbative duality transformations.

If  $B \neq 0$  the lagrangian is not invariant. As it is well known, then the duality transformation is only a symmetry of the equations of motion and not of the lagrangian.

Since  $\tilde{G}_{-A}^{\mu\nu} = \tilde{\mathcal{N}}_{AB} \tilde{\mathcal{F}}_{\mu\nu}^{-B}$  one also has

$$\tilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1} . \quad (2.10)$$

A duality transformation will be a symmetry of the theory if  $\tilde{\mathcal{N}}(\tilde{X}) = \mathcal{N}(\tilde{X})$ , which implies  $\tilde{F}(\tilde{X}) = F(\tilde{X})$ .

Note that  $B \neq 0$  means that the coupling constant  $\tilde{\mathcal{N}}$  is inverted and symmetry transformations with  $B \neq 0$  will be called quantum non perturbative duality symmetries.

The perturbative duality rotations are of the form

$$\begin{pmatrix} A & 0 \\ C & (A^T)^{-1} \end{pmatrix} , \quad A \in GL(r) , \quad A^T C \text{ symmetric} . \quad (2.11)$$

In rigid supersymmetry the tree level symmetries are of the form  $\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}$  while the quantum perturbative monodromy introduces a  $C \neq 0$ .

The general form of the central charge for BPS states in a generic  $N = 2$  rigid theory is given by [1]

$$|Z| = M = |n_{(m)}^A F_A - n_A^{(e)} X^A| , \quad (2.12)$$

where  $n_{(m)}^A, n_A^{(e)}$  denote the values of magnetic and electric charges of the state of mass  $M$ . The above expression is manifestly symplectic covariant provided the vector

$(n_{(m)}^A, n_A^{(e)})$  is also transformed under  $Sp(2r; \mathbb{Z})$ . This equation shows again that a duality symmetry can only be a (perturbative) symmetry if  $B = 0$ , otherwise the vector subspace with  $n_{(m)}^A = 0$  cannot be left invariant.

If the original unbroken gauge group is  $G = SU(r+1)$ , then  $A \in$  Weyl group and  $A^T C$  is the Cartan matrix  $\langle \alpha_i | \alpha_j \rangle$  of  $SU(r+1)$ [2].

Eq. (2.10) shows that  $A + B\mathcal{N}$  has to be invertible in order that the new tensor  $\tilde{\mathcal{N}}$  exists. This is insured by the positive definiteness of  $\text{Im } \mathcal{N}$ , which is the kinetic matrix. Here  $A + B\mathcal{N} = \partial \tilde{X} / \partial X$ , so this implies the invertibility of the mapping  $X \rightarrow \tilde{X}$ . As explained in (2.7), this then also implies the existence of  $\tilde{F}$ . We will see that in local supersymmetry  $\mathcal{N}_{AB} \neq \overline{F}_{AB}$ , so that the existence of  $\tilde{F}$  is not equivalent to the invertibility of  $\text{Im } \mathcal{N}$ , and  $\tilde{F}$  not always exists.

Special coordinates do not give a coordinate independent description of the effective action. A coordinate independent description is obtained by introducing a holomorphic symplectic bundle  $V = (X^A(z), F_A(z))$  and holomorphic  $(1, 0)$  forms on the Kähler manifold[4,1]

$$U_i \equiv \partial_i V = (\partial_i X^A, \partial_i F_A) \quad \text{with } i = 1, \dots, r. \quad (2.13)$$

In rigid special geometry the  $U_i$  satisfy the constraints[4]

$$\begin{aligned} \mathcal{D}_i U_j &= i C_{ijk} g^{k\bar{l}} \overline{U}_{\bar{l}} \\ \partial_i \overline{U}_{\bar{j}} &= 0. \end{aligned} \quad (2.14)$$

Taking then the metric

$$\begin{aligned} g_{i\bar{j}} &= \partial_i \partial_{\bar{j}} K = i(\partial_{\bar{j}} \overline{F}_A \partial_i X^A - \partial_{\bar{j}} \overline{X}^A \partial_i F_A) \\ &= i \partial_i X^A \partial_{\bar{j}} \overline{X}^B (\mathcal{N}_{AB} - \overline{\mathcal{N}}_{AB}), \end{aligned} \quad (2.15)$$

where we used

$$\partial_{\bar{i}} \overline{F}_A = \mathcal{N}_{AB} \partial_{\bar{i}} \overline{X}^B, \quad (2.16)$$

one may derive the tensor  $C_{ijk}$

$$\begin{aligned} C_{ikp} &= \partial_i X^A \mathcal{D}_k \partial_p F_A - \partial_i F_A \mathcal{D}_k \partial_p X^A \\ &= \partial_i X^B (\partial_k \partial_p F_B - \partial_k \partial_p X^A \overline{\mathcal{N}}_{AB}). \end{aligned} \quad (2.17)$$

The integrability conditions on (2.14) yields

$$R_{i\bar{j}k\bar{l}} = -C_{ikp}\overline{C}_{j\bar{l}\bar{p}}g^{p\bar{p}}. \quad (2.18)$$

The Bianchi identities of (2.18) also imply that  $C_{ijk}$  is a holomorphic completely symmetric tensor obeying  $\mathcal{D}_{[i}C_{j]kl} = 0$ .

Note that from (2.17) it also follows

$$C_{ijk} = \partial_i X^A \partial_j X^B \partial_k X^C \partial_A \partial_B \partial_C F, \quad (2.19)$$

which in special coordinates reduces to

$$C_{ABC} = \partial_A \partial_B \partial_C F. \quad (2.20)$$

## 2.2 Symplectic transformations in the fermionic sector

In the total supersymmetric action, the vectors also couple to fermions by terms linear in the field strength. We will first give the general features of the formulation of symplectic transformations in the presence of a fermionic sector, which could even be non-supersymmetric. Afterwards, we will specify the formulae for generic fermionic terms which we encounter in  $N = 2$  lagrangians.

The general form of the Lagrangian, deleting terms which are by themselves symplectic invariant, is

$$\mathcal{L} = -\frac{i}{2}\overline{\mathcal{N}}_{AB}\mathcal{F}^{-A\mu\nu}\mathcal{F}_{\mu\nu}^{-B} - i\mathcal{F}^{-A\mu\nu}H_{A\mu\nu}^- + c.c. + \mathcal{L}_{4f}, \quad (2.21)$$

where  $H_{A\mu\nu}^-$  are quadratic in the fermions, and  $\mathcal{L}_{4f}$  are the quartic terms in fermions. Then

$$G_{A\mu\nu}^- \equiv i\frac{\delta\mathcal{L}}{\delta\mathcal{F}^{-A\mu\nu}} = \overline{\mathcal{N}}_{AB}\mathcal{F}_{\mu\nu}^{-B} + H_{A\mu\nu}^- = G_{bA\mu\nu}^- + H_{A\mu\nu}^-. \quad (2.22)$$

As argued in ref [17], the point where the equations of motions (2.8) are satisfied is an invariant point. Thus, the first term of the action is (omitting the obvious  $A$  indices)

$$\begin{aligned} \mathcal{L}_V &\equiv -\frac{i}{2}\overline{\mathcal{N}}\mathcal{F}_{\mu\nu}^-\mathcal{F}^{-\mu\nu} + c.c. \\ &= -\frac{i}{2}G_{b\mu\nu}^-\mathcal{F}^{\mu\nu} + c.c. \\ &= i\partial^\mu G_{b\mu\nu}^- A^\nu + c.c. \\ &= -i\partial^\mu H_{\mu\nu}^- A^\nu + c.c. - 2\partial^\mu \text{Im } G_{\mu\nu}^- A^\nu \\ &= \frac{i}{2}H_{\mu\nu}^-\mathcal{F}^{-\mu\nu} + c.c. - 2\partial^\mu \text{Im } G_{\mu\nu}^- A^\nu. \end{aligned} \quad (2.23)$$



Therefore

$$\mathcal{L}|_{\frac{\delta \mathcal{L}}{\delta \mathcal{A}}=0} = -\frac{i}{2} H_{\mu\nu}^- \mathcal{F}^{-\mu\nu} + c.c. + \mathcal{L}_{4f} \equiv \mathcal{L}_{inv} , \quad (2.24)$$

which should thus be invariant. The Lagrangian (2.21) is then

$$\mathcal{L} = -\frac{i}{2} \mathcal{F}^{-A\mu\nu} G_{A\mu\nu}^- + c.c. + \mathcal{L}_{inv} . \quad (2.25)$$

Now we suppose  $H_{A\mu\nu}^-$  to be of the form

$$H_{A\mu\nu}^- = (P_{Aa} - \overline{\mathcal{N}}_{AB} Q_a^B) \mathcal{T}_{\mu\nu}^{-a} , \quad (2.26)$$

where  $a$  denotes a new index, whose meaning depends on the model.  $\mathcal{T}_{\mu\nu}^{-a}$  is a tensor not transforming under the symplectic group. Then

$$\begin{aligned} \mathcal{L}_{inv} &= -\frac{i}{2} \mathcal{F}^{-A\mu\nu} (P_{Aa} - \overline{\mathcal{N}}_{AB} Q_a^B) \mathcal{T}_{\mu\nu}^{-a} + c.c. + \mathcal{L}_{4f} \\ &= -\frac{i}{2} (\mathcal{F}^{-A\mu\nu} P_{Aa} - G_{bA}^{-\mu\nu} Q_a^B) \mathcal{T}_{\mu\nu}^{-a} + c.c. + \mathcal{L}_{4f} . \end{aligned} \quad (2.27)$$

Invariance of  $\mathcal{L}_{inv}$  is then guaranteed if  $(Q^A, P_A)$  is a symplectic vector, and  $\mathcal{L}_{4f}$  is constructed as the completion of  $G_b$  to  $G$  in the above formula (plus possible completely invariant terms). These completions are thus

$$\mathcal{L}_{4f} = \frac{i}{2} H_A^{-\mu\nu} Q_a^A \mathcal{T}_{\mu\nu}^{-a} + c.c. + \text{invariant terms} . \quad (2.28)$$

### 2.3 Fermions in $N = 2$ rigid Yang–Mills theory

The coordinate independent description of fermions is given by  $SU(2)$  doublets  $(\lambda^{iI}, \lambda_{\bar{I}}^{\bar{i}})$  where upper and lower  $SU(2)$  indices  $I$  mean positive and negative chiralities respectively [3][26][27]. As such the spinors are symplectic invariant and contravariant world vector fields. The antiselfdual field strength  $\mathcal{F}_{\alpha\beta}^{-A}$  and positive chiralities spinors are in the same  $N = 2$  multiplet, which is, in two component spinor notation,<sup>★</sup>

$$(X^A, \partial_i X^A \lambda_{\alpha}^{iI}, \mathcal{F}_{\alpha\beta}^{-A}) , \quad (2.29)$$

with  $\alpha, \beta \in SL(2, \mathbb{C})$ .

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★  $\mathcal{F}_{\alpha\beta}^{-A}$  is  $\sigma_{\alpha\beta}^{\mu\nu} \mathcal{F}_{\mu\nu}^{-A}$ .

In our application of (2.26) only  $\mathcal{T}$  is dependent on the fermions  $\lambda^{iI}$ , while  $P$  and  $Q$  depend on the scalars  $X^A$ . The index  $a$  is now replaced by  $\bar{i}$ , and we have

$$\begin{aligned} Q_{\bar{i}}^A &= \partial_{\bar{i}} \bar{X}^A ; & P_{A\bar{i}} &= \partial_{\bar{i}} \bar{F}_A \\ \mathcal{T}_{\alpha\beta}^{\bar{i}} &= k g^{\bar{i}j} C_{j\bar{k}p} \lambda_{\alpha}^{kI} \lambda_{\beta}^{pJ} \epsilon_{IJ} , \end{aligned} \quad (2.30)$$

where  $k$  is a constant to be determined by supersymmetry. Then

$$H_{-A}^{\alpha\beta} = k \partial_{\bar{i}} \bar{X}^B (\mathcal{N}_{BA} - \bar{\mathcal{N}}_{BA}) g^{\bar{i}j} C_{j\bar{k}p} \lambda^{\alpha kI} \lambda^{\beta pJ} \epsilon_{IJ} . \quad (2.31)$$

This yields

$$\begin{aligned} \mathcal{L}_{\text{Pauli}} &= -i(\mathcal{N} - \bar{\mathcal{N}})_{AB} \partial_{\bar{i}} \bar{X}^A \mathcal{T}_{\alpha\beta}^{\bar{i}} \mathcal{F}^{B\alpha\beta} + c.c. \\ \mathcal{L}_{4f} &= \frac{i}{2} \partial_{\bar{i}} \bar{X}^A \partial_{\bar{j}} \bar{X}^B (\bar{\mathcal{N}}_{AB} - \mathcal{N}_{AB}) \mathcal{T}_{\alpha\beta}^{\bar{i}} \mathcal{T}^{\bar{j}\alpha\beta} + c.c. + \text{invariant terms} , \end{aligned} \quad (2.32)$$

in agreement with Cremmer et al. [30].

In special coordinates, setting  $\lambda_{\alpha}^{i1} = \chi_{\alpha}^i$ ,  $\lambda_{\alpha}^{i2} = \lambda_{\alpha}^i$ , the Pauli term reduces to

$$\mathcal{L}_{\text{Pauli}} = -k \partial_A \partial_B \partial_C F (\chi_{\alpha}^A \lambda_{\beta}^B - \lambda_{\alpha}^A \chi_{\beta}^B) \mathcal{F}^{-C\alpha\beta} + c.c. , \quad (2.33)$$

in agreement with the standard  $N = 1$  supersymmetric action with  $f_{AB} = F_{AB}$  [30]. We see from (2.32) that in rigid supersymmetry the physical meaning of  $C_{ijk}$  is that of an anomalous magnetic moment. Note that  $C_{ijk}$  vanishes at tree-level and it is  $\sim \frac{1}{\langle X \rangle}$  at one loop-level as it must be [19][20][1]. It is obviously singular at  $\langle X \rangle = 0$ . In the  $SU(2)$  quantum theory [1], the  $SU(2)$  symmetry is not restored at  $X = 0$ , and then one rather expects such terms to behave as  $\frac{c_0}{\Lambda}$  where  $c_0$  is a dimensionless number. The vanishing at tree-level of both Pauli terms and the corresponding four fermions terms is consistent with renormalizability arguments.

The other fermionic terms which are already duality invariant read

$$\lambda_{\alpha}^{iI} \lambda_{\beta}^{kJ} \epsilon^{\alpha\beta} \bar{\lambda}_{\dot{\alpha}I}^{\bar{j}} \bar{\lambda}_{\dot{\beta}J}^{\bar{k}} \epsilon^{\dot{\alpha}\dot{\beta}} R_{\bar{i}\bar{j}k\bar{l}} \quad (2.34)$$

and

$$\mathcal{D}_i C_{jlm} \lambda_{\alpha}^{iI} \lambda_{\beta}^{jK} \epsilon^{\alpha\beta} \lambda_{\gamma}^{lJ} \lambda_{\delta}^{mL} \epsilon^{\gamma\delta} \epsilon_{IJ} \epsilon_{KL} . \quad (2.35)$$

Note that, because of eq. (2.18), all couplings in the lagrangian are expressed through the tensors  $C_{ijk}$ .

From a tensor calculus point of view, all quartic terms but the last come from the equations of motion of the  $Y_{IJ}^i$  auxiliary field triplet[3].

## 2.4 Positivity and monodromies

Let us consider a submanifold  $\mathcal{M}_r$  of the moduli space of a Riemann surface of genus  $r$  such that its tangent space is isomorphic to the Hodge bundle. In particular the dimension of  $\mathcal{M}_r$  is equal to the genus  $r$  of the Riemann surface  $\mathcal{C}_r$ <sup>†</sup>. In this case, decomposing an abelian differential in terms of the  $2r$  harmonic forms dual to the canonical basis of cycles, we have

$$\begin{aligned} \omega &= X^A(z^i)\alpha_A + F_A(z^i)\beta^A \quad A, i = 1, \dots, r \\ \int \alpha_A \wedge \beta^B &= \delta_A^B \quad , \quad \int \alpha_A \wedge \alpha_B = \int \beta_A \wedge \beta_B = 0 \quad , \end{aligned} \quad (2.36)$$

where  $z^i$  are coordinates on the moduli space submanifold, and

$$\partial_i \omega = \partial_i X^A \alpha_A + \partial_i F_A \beta^A \quad . \quad (2.37)$$

Then the metric, given by the norm

$$g_{i\bar{j}} = i \int \partial_i \omega \wedge \partial_{\bar{j}} \bar{\omega} = i \partial_i \partial_{\bar{j}} \int \omega \wedge \bar{\omega} \quad (2.38)$$

is manifestly positive. Using eqs. (2.36), (2.37) we find

$$g_{i\bar{j}} = i \partial_i \partial_{\bar{j}} (\bar{F}_A X^A - \bar{X}^A F_A)$$

which coincides with the metric of  $N = 2$  rigid special geometry (2.15) [1,4].

Formula (2.37) implies by supersymmetry a similar expansion for the full multiplet (2.29). For the upper component  $\mathcal{F}_{\mu\nu}^{-A}$  we get a self dual three form

$$w = \mathcal{F}^A \alpha_A + G_A \beta^A \quad (2.39)$$

on  $\mathbf{R}_4 \times \mathcal{C}_r$  when (2.8) hold. We observe that an  $N = 2$ ,  $4D$  abelian vector multiplet can be obtained from dimensional reduction from six dimensions either of a vector multiplet or of a tensor multiplet containing a self-dual field strength. This remarkable coincidence actually suggests a physical picture for the characterization of this subclass

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<sup>†</sup> We are aware of the fact that to find an intrinsic characterization of such an algebraic locus is far from obvious. We thank D. Dubrovin, D. Franco, P. Fré and C. Reina for clarifying discussions on this point.

$\mathcal{C}_r$  of Riemann surfaces. Namely, they should appear in the compactification on  $\mathbf{R}_4 \times \mathcal{C}_r$  of  $N = 1$  six-dimensional theory of a self interacting tensor multiplet.

As shown in ref. [4], the Picard–Fuchs equations for  $\mathcal{C}_r$  have a general form dictated by the differential constraints of rigid special geometry. A general proposal for  $\mathcal{C}_r$  has been given in [2] and can be used to write down the Picard–Fuchs equations for the periods and to determine their monodromies. Such proposal can be checked by comparing the explicit form of the Picard–Fuchs equations with their general form given by rigid special geometry.

In the one parameter case ( $G = SU(2)$ ), where  $\mathcal{C}_1$  is given by the elliptic curve of ref. [1], the special geometry equations reduce to one ordinary second order equation

$$\left(\frac{d}{dz} + \widehat{\Gamma}\right)C^{-1}\left(\frac{d}{dz} - \widehat{\Gamma}\right)U = 0 \quad (2.40)$$

where  $\widehat{\Gamma} = \frac{d}{dz} \log e$ ,  $e = \frac{dX}{dz}$  and  $C$  is the 3-tensor appearing in (2.14). This agrees with the Picard–Fuchs equations derived from  $\mathcal{C}_1$ . The general solution of this equation is [4]

$$U = \left(e, e \frac{d^2 F}{dX^2}\right) \quad , \quad (2.41)$$

with  $\tau = \frac{d^2 F}{dX^2}$  being the uniformizing variable for which the differential equation reduces to  $\frac{d^2}{d\tau^2}(\ ) = 0$ .

### 3 Coupling to gravity

#### 3.1 Special geometry and symplectic transformations

The coupling to gravity modifies the constraints of rigid special geometry because of the introduction of a  $U(1)$  connection due to the  $U(1)$  Kähler–Hodge structure of moduli space. For  $n$  vector multiplets one introduces  $2(n+1)$  covariantly holomorphic sections [26,23,27,29]

$$V = (L^\Lambda, M_\Lambda) \quad (\Lambda = 0, \dots, n) \quad , \quad (3.1)$$

where 0 is the graviphoton index.

The new differential constraints of special geometry are

$$\begin{aligned}
U_i &\equiv (\mathcal{D}_i L^\Lambda, \mathcal{D}_i M_\Lambda) = (f_i^\Lambda, h_{i\Lambda}) \\
\mathcal{D}_i U_j &= i C_{ijk} g^{k\bar{l}} \overline{U}_{\bar{l}} \\
\mathcal{D}_i \overline{U}_{\bar{j}} &= g_{i\bar{j}} \overline{V} \\
\mathcal{D}_i \overline{V} &= 0 ,
\end{aligned} \tag{3.2}$$

where now  $\mathcal{D}_i$  is the covariant derivative with respect to the usual Levi-Civita connection and the Kähler connection  $\partial_i K$ . That is, under  $K \rightarrow K + f + \bar{f}$  a generic field  $\psi^i$  which under  $U(1)$  transforms as  $\psi^i \rightarrow e^{-(\frac{p}{2}f + \frac{\bar{p}}{2}\bar{f})} \psi^i$  has the following covariant derivative

$$\mathcal{D}_i \psi^j = \partial_i \psi^j + \Gamma_{ik}^j \psi^k + \frac{p}{2} \partial_i K \psi^j , \tag{3.3}$$

and analogously for  $\mathcal{D}_{\bar{i}}$  with  $p \rightarrow \bar{p}$ . This  $U(1)$  is related to the  $U(1)$  in the  $N = 2$  superconformal group, and the weights for all the fields were determined in [31] ( $\bar{p} = c$ ). In our notations,  $(L^\Lambda, M_\Lambda)$  have been given conventionally weights  $p = -\bar{p} = 1$ .

Since  $L^\Lambda, M_\Lambda$  are covariantly holomorphic, it is convenient to introduce holomorphic sections  $X^\Lambda = e^{-K/2} L^\Lambda$ ,  $F_\Lambda = e^{-K/2} M_\Lambda$ .

The Kähler potential is fixed by the condition[3][26]

$$i(\overline{L}^\Lambda M_\Lambda - L^\Lambda \overline{M}_\Lambda) = 1 \tag{3.4}$$

to be

$$K = -\log i(\overline{X}^\Lambda F_\Lambda - X^\Lambda \overline{F}_\Lambda) . \tag{3.5}$$

As it is well known[3][32], the differential constraints (3.2) can in general be solved in terms of a holomorphic function homogeneous of degree two  $F(X)$ . However, as we will see in the sequel, there exist particular symplectic sections for which such prepotential  $F$  does not exist. In particular this is the case appearing in the effective theory of the  $N = 2$  heterotic string. For this reason it is convenient to have the fundamental formulas of special geometry written in a way independent of the existence of  $F$ .

First of all we note that quite generally we may write

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma} L^\Sigma ; \quad h_{\Lambda i} = \overline{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma . \tag{3.6}$$

From (3.6) we can define the two  $(n+1) \times (n+1)$  matrices

$$h_{\Lambda\bar{I}} = (h_{\Lambda\bar{0}} \equiv M_\Lambda, h_{\Lambda\bar{i}}) \quad , \quad f_{\bar{I}}^\Lambda = (f_0^\Lambda \equiv L^\Lambda, f_i^\Lambda) \quad (3.7)$$

to obtain an explicit expression for  $\mathcal{N}_{\Lambda\Sigma}$  in terms of  $(L^\Lambda, M_\Lambda)$  as

$$\mathcal{N}_{\Lambda\Sigma} = h_{\Lambda\bar{I}}(f^{-1})_{\Sigma}^{\bar{I}} \quad . \quad (3.8)$$

Note that  $h_{\Lambda\bar{I}}, f_{\bar{I}}^\Sigma$  are invertible matrices and the above expression implies the transformation law (2.10).

When  $F$  exists,  $\mathcal{N}_{\Lambda\Sigma}$  has the form[3][27]

$$\mathcal{N}_{\Lambda\Sigma} = \overline{F}_{\Lambda\Sigma} + 2i \frac{(\text{Im } F_{\Lambda\Gamma})(\text{Im } F_{\Sigma\Pi})L^\Gamma L^\Pi}{(\text{Im } F_{\Xi\Omega})L^\Xi L^\Omega} \quad , \quad (3.9)$$

which turns out to be the coupling matrix appearing in the kinetic term of the vector fields. However, as we show below, (3.6) are symplectic covariant and therefore they always hold even in some specific coordinate system in which  $F$  does not exist.

In the same way as in the rigid case, from eqs. (3.2) and (3.4) we find

$$g_{i\bar{j}} = i(f_i^\Lambda \bar{h}_{\bar{j}\Lambda} - h_{i\Lambda} \bar{f}_{\bar{j}}^\Lambda) = i(\mathcal{N}_{\Lambda\Sigma} - \overline{\mathcal{N}}_{\Lambda\Sigma}) f_i^\Lambda \bar{f}_{\bar{j}}^\Sigma \quad (3.10)$$

$$C_{ijk} = f_i^\Lambda \mathcal{D}_j h_{k\Lambda} - h_{i\Lambda} \mathcal{D}_j f_k^\Lambda = f_i^\Lambda \partial_j \overline{\mathcal{N}}_{\Lambda\Sigma} f_k^\Sigma \quad , \quad (3.11)$$

which are symplectic invariant. (Note that  $\mathcal{N}_{\Lambda\Sigma}$  has zero Kähler weight).

Furthermore, the integrability conditions (3.2) give[3][26][25][23][29][27]

$$R_{i\bar{j}l\bar{k}} = g_{i\bar{j}} g_{l\bar{k}} + g_{i\bar{k}} g_{l\bar{j}} - C_{ilp} C_{\bar{j}\bar{k}\bar{p}} g^{p\bar{p}} \quad , \quad (3.12)$$

replacing eq. (2.6).

Here  $C_{ilp}$  is a covariantly holomorphic tensor of weight  $p = -\bar{p} = 2$ ,

$$\mathcal{D}_{\bar{l}} C_{ijk} = \partial_{\bar{l}} C_{ijk} - \partial_{\bar{l}} K C_{ijk} = 0 \quad , \quad (3.13)$$

which implies  $\partial_{\bar{l}} W_{ijk} = 0$  with  $C_{ijk} = e^K W_{ijk}$ .

Some additional consequences of the previous formulae are the following: from  $\mathcal{D}_i F_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma} \mathcal{D}_i X^\Sigma$ , applying  $\mathcal{D}_{\bar{j}}$  to both sides we also find

$$\mathcal{D}_{\bar{j}} \mathcal{D}_i F_\Lambda = \partial_{\bar{j}} \overline{\mathcal{N}}_{\Lambda\Sigma} \mathcal{D}_i X^\Sigma + \overline{\mathcal{N}}_{\Lambda\Sigma} \mathcal{D}_{\bar{j}} \mathcal{D}_i X^\Sigma \quad , \quad (3.14)$$

which implies, using the third line of (3.2),

$$(F_\Lambda - \overline{\mathcal{N}}_{\Lambda\Sigma} X^\Sigma) g_{i\overline{j}} = \partial_{\overline{j}} \overline{\mathcal{N}}_{\Lambda\Sigma} \mathcal{D}_i X^\Sigma . \quad (3.15)$$

Note that the left-hand side of (3.15) defines the graviphoton projector

$$T_\Lambda = M_\Lambda - \overline{\mathcal{N}}_{\Lambda\Sigma} L^\Sigma . \quad (3.16)$$

From the first of equations (3.6) it also follows that

$$\partial_{\overline{i}} \mathcal{N}_{\Lambda\Sigma} L^\Sigma = 0 \quad , \quad h_{i\Lambda} = \mathcal{N}_{\Lambda\Sigma} f_i^\Sigma + \partial_i \mathcal{N}_{\Lambda\Sigma} L^\Sigma \quad (3.17)$$

and therefore

$$\partial_i \mathcal{N}_{\Lambda\Sigma} L^\Sigma = (\overline{\mathcal{N}}_{\Lambda\Sigma} - \mathcal{N}_{\Lambda\Sigma}) f_i^\Sigma \quad (3.18)$$

by contraction with  $f_j^\Lambda$  we get

$$f_j^\Lambda \partial_i \mathcal{N}_{\Lambda\Sigma} L^\Sigma = i g_{i\overline{j}} . \quad (3.19)$$

Taking the complex conjugate of (3.19) and using (3.15) it follows that

$$T_\Lambda \overline{L}^\Lambda = -i . \quad (3.20)$$

which is nothing but (3.4). An alternative form for the Kähler potential is

$$K = -\log \ i(\mathcal{N}_{\Lambda\Sigma} - \overline{\mathcal{N}}_{\Lambda\Sigma}) X^\Lambda \overline{X}^\Sigma . \quad (3.21)$$

Duality transformations are now in  $Sp(2n+2, \mathbb{Z})$  and act on  $X^\Lambda, F_\Lambda$  as in the rigid case. The symplectic action on  $(L^\Lambda, M_\Lambda)$  (or  $(X^\Lambda, F_\Lambda)$ ) is

$$\begin{pmatrix} L \\ M \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} L \\ M \end{pmatrix} = \mathcal{S} \begin{pmatrix} L \\ M \end{pmatrix} \quad \mathcal{S} \in Sp(2n+2, \mathbb{Z}) . \quad (3.22)$$

Then it follows, because of eq. (3.2) and (3.6),

$$\begin{pmatrix} f_i^\Lambda \\ h_{i\Lambda} \end{pmatrix}' = \begin{pmatrix} A & B\overline{\mathcal{N}} \\ C & D\overline{\mathcal{N}} \end{pmatrix} \begin{pmatrix} f_i^\Lambda \\ f_i^\Lambda \end{pmatrix} , \quad (3.23)$$

which implies again (2.10). These two transformations laws imply the covariance of (3.6).

The symplectic action on  $\mathcal{F}_{\mu\nu}^{+\Lambda}, G_{+\Lambda}^{\mu\nu}$  is the same as on  $(L^\Lambda, M_\Lambda)$ , so eq. (2.8) is unchanged. Therefore the discussion of the previous section on perturbative and non perturbative duality transformations in the rigid case remains unchanged when gravity is turned on.

When the sections  $(X^\Lambda, F_\Lambda)$  are chosen in such a way that a function  $F$  exists<sup>★</sup>, from (3.4) and the degree two homogeneity of  $F$  it follows that [26][27]

$$\text{Im } F_{\Lambda\Sigma} L^\Lambda \overline{f}_i^\Sigma = 0 , \quad (3.24)$$

so that the second of eq. (3.6) becomes  $h_{i\Lambda} = F_{\Lambda\Sigma} f_i^\Sigma$ . Furthermore from (3.11) and (3.24) it also follows

$$e^{K/2} C_{ijk} = f_i^\Lambda f_j^\Gamma f_k^\Sigma F_{\Lambda\Gamma\Sigma} . \quad (3.25)$$

By the same token, we have

$$\begin{pmatrix} f_i^\Lambda \\ h_{i\Lambda} \end{pmatrix}' = \begin{pmatrix} A & B\mathcal{F} \\ C & D\mathcal{F} \end{pmatrix} \begin{pmatrix} f_i^\Lambda \\ f_i^\Lambda \end{pmatrix} , \quad (3.26)$$

where  $\mathcal{F} = F_{\Lambda\Sigma}$ . Note that in these cases

$$\begin{aligned} 2\tilde{F}(\tilde{X}) &= \tilde{F}_\Lambda \tilde{X}^\Lambda = \\ &= 2F + 2X^\Lambda (C^T B)_\Lambda^\Sigma F_\Sigma + X^\Lambda (C^T A)_{\Lambda\Sigma} X^\Sigma + F_\Lambda (D^T B)^{\Lambda\Sigma} F_\Sigma . \end{aligned} \quad (3.27)$$

Note also that the homogeneity of  $F$  implies

$$\tilde{X} = (A + B\mathcal{F})X , \quad (3.28)$$

where  $\mathcal{F} = F_{\Lambda\Sigma}$  and

$$\tilde{F} = (C + D\mathcal{F})X . \quad (3.29)$$

Special coordinates in supergravity are defined by  $t^\Lambda = X^\Lambda/X^0$  since we now have a set of  $n+1$  homogeneous coordinates. If we assume that  $\mathcal{D}_i(\frac{X^\Lambda}{X^0})$  is an invertible matrix, then we may choose a frame for which  $\partial_i(\frac{X^\Lambda}{X^0}) = \delta_i^\Lambda$ . This is possible only if  $X^\Lambda$  are unconstrained variables and so  $F_\Lambda = F_\Lambda(X)$ , which implies  $F_\Lambda = \partial_\Lambda F(X)$  with  $F$  homogeneous of degree 2.

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★ A résumé of the duality transformations for this case, including the supergravity corrections has been given in appendix C of [32].



We now discuss the possible non-existence of  $F(X)$ . If we start with some special coordinates  $X^\Lambda, F_\Lambda(X)$ , it is possible that in the new basis the  $\tilde{X}^\Lambda$  are not good special coordinates in the sense that the mapping  $X \rightarrow \tilde{X}$  is not invertible. This happens whenever the  $(n+1) \times (n+1)$  matrix  $A + B\mathcal{F}$  is not invertible (its determinant vanishes). This does not mean that  $\tilde{X}, \tilde{F}$  are not good symplectic sections since the symplectic matrix  $\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  is always invertible. It simply means that  $\tilde{F}_\Lambda \neq \tilde{F}_\Lambda(\tilde{X})$  and therefore a prepotential  $\tilde{F}(\tilde{X})$  does not exist. However our formulation of special geometry never explicitly used the fact that  $F_\Lambda$  be a functional of the  $X$ 's and indeed the quantities  $(X^\Lambda, F_\Lambda)$ ,  $(f_i^\Lambda, h_{i\Lambda})$ ,  $\mathcal{N}_{\Lambda\Sigma}$  and  $C_{ijk}, g_{i\bar{j}}$  are well defined for any choice of the symplectic sections  $(X^\Lambda, F_\Lambda)$  since they are symplectic invariant or covariant. For example, to compute the “gauge coupling”  $\tilde{\mathcal{N}}$  in such a basis  $(\tilde{X}^\Lambda, \tilde{F}_\Lambda)$  one uses the formula

$$\tilde{\mathcal{N}}(\tilde{X}, \tilde{F}) = (C + D\mathcal{N}(X))(A + B\mathcal{N}(X))^{-1}, \quad (3.30)$$

and expresses the  $X = X(\tilde{X}, \tilde{F})$  by using the fact that the symplectic mapping can be inverted. All other quantities can be computed in this way.

We will see the relevance of this observation in the sequel, while discussing low energy effective action of  $N = 2$  heterotic string. A simple example is the following. Consider  $F = iX^0X^1$ , leading to

$$\mathcal{N} = \begin{pmatrix} i\frac{X^1}{X^0} & 0 \\ 0 & i\frac{X^0}{X^1} \end{pmatrix}. \quad (3.31)$$

This appears in the  $N = 2$  reduction of pure  $N = 4$  supergravity in the so-called  $SO(4)$  formulation [33]. Consider now the symplectic mapping defined by

$$A = D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad C = -B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.32)$$

Then the transformation is

$$\begin{aligned} \tilde{X}^0 &= X^0 & \tilde{X}^1 &= -F_1 \\ \tilde{F}_0 &= F_0 & \tilde{F}_1 &= X^1. \end{aligned} \quad (3.33)$$

Using in the first line  $F_1 = iX^0$  would lead to a non-invertible mapping  $X \rightarrow \tilde{X}$ , and using (3.27) would lead to  $\tilde{F} = 0$ . One observes also that  $A + B\mathcal{F}$  is non-invertible. However,  $A + B\mathcal{N}$  is invertible, and one obtains  $\tilde{\mathcal{N}} = iX^1(X^0)^{-1}\mathbb{1} = i\tilde{F}_1(\tilde{X}^0)^{-1}\mathbb{1}$ . This form appears in the  $N = 2$  reduction of the  $SU(4)$  formulation of pure  $N = 4$  supergravity [34]. These two forms of the  $N = 2$  reduced action and the duality transformation have been studied in [35] to relate electric and magnetic charges of black holes.

### 3.2 The fermionic sector

As far as the fermions are concerned, the vector  $N = 2$  multiplet is now

$$(L^\Lambda, f_i^\Lambda \lambda_\alpha^{iI}, \mathcal{F}_{\alpha\beta}^{-\Lambda}) . \quad (3.34)$$

The tensor  $\mathcal{T}_{\alpha\beta}^{\bar{i}}$  is still the same as in (2.30), and

$$Q_{\bar{i}}^\Lambda = \mathcal{D}_{\bar{i}} \bar{L}^\Lambda ; \quad P_{\Lambda\bar{i}} = \mathcal{D}_{\bar{i}} \bar{M}_\Lambda . \quad (3.35)$$

Correspondingly, the gaugino Pauli terms have the form

$$i(\mathcal{D}_{\bar{i}} \bar{L}^\Lambda G_{b-\Lambda}^{\alpha\beta} - \mathcal{D}_{\bar{i}} \bar{M}_\Lambda \mathcal{F}^{-\Lambda\alpha\beta}) \mathcal{T}_{\alpha\beta}^{\bar{i}} , \quad (3.36)$$

quite analogous to eq. (2.32).

Gravitino Pauli and quartic terms [3][30][27] are defined by the formulas (2.21) and (2.28) with<sup>★</sup>

$$\begin{aligned} Q^\Lambda &= L^\Lambda ; & P_\Lambda &= M_\Lambda \\ \mathcal{T}^{\mu\nu} &= k_1 \bar{\psi}_\rho^I \psi_\sigma^J \epsilon_{IJ} \epsilon^{\mu\nu\rho\sigma} \end{aligned} \quad (3.37)$$

for the purely gravitino terms, in which case the index  $a$  of the general treatment is obsolete. For the mixed gaugino–gravitino Pauli terms we use

$$\begin{aligned} Q_{\bar{i}}^\Lambda &= \mathcal{D}_{\bar{i}} \bar{L}^\Lambda ; & P_{\Lambda\bar{i}} &= \mathcal{D}_{\bar{i}} \bar{M}_\Lambda \\ \mathcal{T}_{\alpha\beta}^{\bar{i}} &= k_2 \bar{\lambda}_I^{\bar{i}} \gamma_\rho \psi_{\sigma J} \epsilon^{IJ} \epsilon^{\mu\nu\rho\sigma} , \end{aligned} \quad (3.38)$$

and the index  $\bar{i}$  plays again the role of  $a$ . The constants  $k$ ,  $k_1$  and  $k_2$  should also be fixed by supersymmetry. So, as before, the unique quartic terms are generated by requiring duality invariance of the action. Of course many of these terms are absent in  $N = 1$  [30] theories because of the absence of the second gravitino. This is one of the differences between rigid supersymmetry and local supersymmetry. What happens is that in  $N = 2$  supergravity, one introduces an extra  $(\frac{3}{2}, 1)$  multiplet, with respect to the  $N = 1$  case. This has the effect of having extra auxiliary fields in the supergravity multiplet[36]

$$\mathcal{V}_{J\mu}^I , \quad A_\mu , \quad T_{\mu\nu}^- , \quad D \quad (3.39)$$

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★ The Kähler weights of the fermions are  $p = -\bar{p} = \frac{1}{2}$  for  $\psi_{\mu I}$ , and  $p = -\bar{p} = -\frac{1}{2}$  for  $\lambda^{iI}$ . The scalars and the fermions of the hypermultiplets, not discussed here, have respectively Kähler weights  $p = \bar{p} = 0$  and  $p = -\bar{p} = -\frac{1}{2}$ .

other than the matter auxiliary field of the vector multiplet  $Y^{IJ}$  (traceless, real, symmetric in  $IJ$ ),  $i, j = 1, 2$ , *i.e.* a real  $SU(2)$  triplet. The meaning of the auxiliary fields is straightforward. The  $Y$ 's correspond to the three auxiliary fields of a  $N = 1$  vector multiplet and a chiral multiplet. The  $D$  auxiliary field gives the equation (3.4) (i.e (3.5)),  $T_{\mu\nu}^-$  is the graviphoton (symplectic invariant) combination of the gauge fields  $T_{\mu\nu}^- = T_\Lambda \mathcal{F}_{\mu\nu}^{-\Lambda}$ , and  $\mathcal{V}_{J\mu}^I$ ,  $A_\mu$  are the composite  $SU(2)$  and  $U(1)$  connections of the quaternionic manifold and Kähler–Hodge manifold respectively. Note that comparison between  $N = 1$  and  $N = 2$  theories shows that the spinors  $\chi^i$  of the scalar multiplet and  $\lambda^\Sigma$  of the vector multiplet of the  $N = 1$  theory are related to the doublet  $\lambda^{iI}$  of the  $N = 2$  theory by

$$\chi^i = \lambda^{i1} \quad , \lambda^\Sigma = f_i^\Sigma \lambda^{i2} \quad . \quad (3.40)$$

### 3.3 The three–form cohomology

We recall that special geometry in  $N = 2$  supergravity, unlike rigid special geometry, is suitable for three–form cohomology for Calabi–Yau manifolds. Let's define a holomorphic three–form [25,23]

$$\Omega = X^\Lambda \alpha_\Lambda + F_\Lambda \beta^\Lambda \quad (3.41)$$

where  $\alpha_\Lambda, \beta^\Lambda$  is a  $2n + 2$  dimensional cohomology basis dual to the  $2n + 2$  homology cycles ( $n = h_{21}$ ).  $\Omega$  is a holomorphic section of a line bundle. Then it follows that if one defines

$$e^{-K} = i \int \Omega \wedge \overline{\Omega} > 0 \quad (3.42)$$

then

$$g_{i\overline{j}} = \frac{-i \int \mathcal{D}_i \Omega \wedge \mathcal{D}_{\overline{j}} \overline{\Omega}}{i \int \Omega \wedge \overline{\Omega}} = -\partial_i \partial_{\overline{j}} \log i \int \Omega \wedge \overline{\Omega} > 0 \quad . \quad (3.43)$$

The  $(2n + 2)$  three–forms  $\mathcal{D}_i \Omega, \mathcal{D}_{\overline{i}} \overline{\Omega}, \Omega, \overline{\Omega}$  with the cohomology basis  $(\alpha_\Lambda, \beta^\Lambda)$  correspond to the decomposition

$$H^3(\mathbf{R}) = H^{(2,1)}(\mathbb{C}) + H^{(1,2)}(\mathbb{C}) + H^{(3,0)}(\mathbb{C}) + H^{(0,3)}(\mathbb{C}) \quad . \quad (3.44)$$

Note that since  $\Omega = (X^\Lambda, F_\Lambda)$ , then  $\mathcal{D}_i \Omega = (\mathcal{D}_i X^\Lambda, \mathcal{D}_i F_\Lambda)$ , with  $f_i^\Lambda = e^{\frac{K}{2}} \mathcal{D}_i X^\Lambda$ ,  $h_{i\Lambda} = e^{\frac{K}{2}} \mathcal{D}_i F_\Lambda$ . The relations

$$\int \Omega \wedge \Omega = \int \Omega \wedge \mathcal{D}_i \overline{\Omega} = 0 \quad (3.45)$$

are obvious since  $\mathcal{D}_i\Omega = \partial_i\Omega - \frac{1}{(\Omega, \overline{\Omega})}(\partial_i\Omega, \overline{\Omega})\Omega$ . However the relation

$$\int \Omega \wedge \mathcal{D}_i\Omega = 0 , \quad (3.46)$$

which is suitable for three-form cohomology, implies

$$\int \Omega \wedge \partial_i\Omega = 0 , \quad (3.47)$$

*i.e.*

$$\partial_i X^\Lambda F_\Lambda - \partial_i F_\Lambda X^\Lambda = 0 \quad (3.48)$$

for any choice of the symplectic section. Eq. (3.48) is equivalent to

$$X^\Lambda \mathcal{D}_i F_\Lambda - \mathcal{D}_i X^\Lambda F_\Lambda = 0 . \quad (3.49)$$

### 3.4 Duality transformations in $N = 1$ locally supersymmetric Yang–Mills theories

In  $N = 1$  super Yang–Mills theories coupled to supergravity [30], duality transformations are implemented as follows. Define the symplectic  $Sp(2r)$  vectors

$$\begin{aligned} \mathcal{V} &= (\mathcal{F}_{\mu\nu}^{-A}, G_A^{-\mu\nu} = i \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\mu\nu}^{-A}}) \\ \mathcal{U}_\alpha &= (\lambda_\alpha^A, f_{AB}(z) \lambda_\alpha^B) \end{aligned} \quad (3.50)$$

where  $(\lambda^A, \mathcal{F}_{\mu\nu}^{-A})$  is the vector field strength multiplet and  $f_{AB}(z)$  is the holomorphic coupling introduced in [30]<sup>★</sup>, which depends on the scalars of chiral multiplets, and which plays here the role of  $\overline{\mathcal{N}}_{AB}$  in the general treatment of sections 2.1 and 2.2. Then the  $N = 1$  supergravity lagrangian is invariant under the symplectic transformations

$$\mathcal{V} \rightarrow \mathcal{S}\mathcal{V} , \quad \mathcal{U} \rightarrow \mathcal{S}\mathcal{U} , \quad f \rightarrow (C + Df)(A + Bf)^{-1} , \quad \mathcal{S} \in Sp(2, r; \mathbf{R}) . \quad (3.51)$$

This is best seen using the  $N = 1$  tensor calculus (or superfield) notation of ref. [30]. The part of the action which contains the field strength chiral multiplet

$$W_\alpha^A = T(\mathcal{D}_\alpha V^A) , \quad (3.52)$$

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★ We replaced the  $f$  in [30] by  $2if$ .

where  $T$  is the generalisation of to local supersymmetry of the chiral projection  $\overline{D}\overline{D}$  (similar to the operation obtaining kinetic multiplets introduced in [37]), can be written in first order form by introducing an unconstrained chiral multiplet  $W_\alpha^A$  and a (vector) real lagrangian multiplier  $U_A$  ( $f_{AB}$  is a chiral superfield)

$$4\text{Im } W_\alpha^A \mathcal{D}_\beta U_A \epsilon^{\alpha\beta} |_D + i f_{AB}(z) W_\alpha^A W_\beta^B \epsilon^{\alpha\beta} |_F . \quad (3.53)$$

Variation with respect to  $U_A$  yields the Bianchi identity

$$\mathcal{D}^\alpha W_\alpha^A = \overline{\mathcal{D}}_{\dot{\alpha}} \overline{W}^{\dot{\alpha}A} , \quad (3.54)$$

which is solved by

$$W_\alpha^A = T(\mathcal{D}_\alpha V^A) , \quad (3.55)$$

which leads to the original form of the action. The dual form of the theory is obtained, in a manner analogous to the rigid case [1], by varying the same lagrangian with respect to  $W_\alpha^A$ . Defining  $W_{\alpha A}^{(D)} \equiv T(\mathcal{D}_\alpha U_A)$ , and using the fact that the first term in (3.53) can also be written as  $-2i W_\alpha^A W_{\beta B}^{(D)} \epsilon^{\alpha\beta} |_F$ , yields

$$W_\alpha^A = (f^{-1})^{AB} W_{\alpha B}^{(D)} , \quad (3.56)$$

which implies the Bianchi identity also for  $W^{(D)}$ . The dual lagrangian is

$$\mathcal{L}^D = -i(f^{-1})^{AB} W_{\alpha A}^{(D)} W_{\beta B}^{(D)} \epsilon^{\alpha\beta} |_F . \quad (3.57)$$

This realises the symplectic transformation of (3.51) with  $B = -C = \mathbb{1}$  and  $A = D = 0$ .

A duality rotation is a symmetry if for some coordinate changes  $z \rightarrow \tilde{z}$  ( $z$  is the first component of a chiral multiplet)

$$\tilde{f}_{AB}(\tilde{z}) = f_{AB}(\tilde{z}) \quad (3.58)$$

and the superpotential  $W$  is a symplectic invariant section of a Hodge bundle, *i.e.*

$$\| \widetilde{W}(\tilde{z}) \|^2 = \| W(\tilde{z}) \|^2 , \quad (3.59)$$

where  $\| W(z) \|^2 = |W(z)|^2 e^K \equiv e^G$ . In component form, we can exhibit the symplectic invariance of the gaugino kinetic term and the Pauli terms by noticing that they can be written as

$$\begin{aligned} e^{-1} \mathcal{L}_{\text{kin}}(\lambda, \bar{\lambda}) &= i \mathcal{U}_\alpha \Omega(\sigma^\mu)^{\alpha\dot{\alpha}} \mathcal{D}_\mu \bar{\mathcal{U}}_{\dot{\alpha}} \\ e^{-1} \mathcal{L}_{\text{Pauli}}(\psi, \lambda) &= \text{Im} (\bar{\mathcal{U}}_{\dot{\alpha}} \Omega(\sigma^\mu)^{\dot{\alpha}\beta} \mathcal{V}_{b\beta\gamma} \psi_\mu^\gamma) \\ e^{-1} \mathcal{L}_{\text{Pauli}}(\chi, \lambda) &= \text{Im} (\partial_i f_{AB} \lambda_\alpha^A \bar{\chi}_\beta^i \mathcal{F}^{-B\alpha\beta}) \end{aligned} \quad (3.60)$$

where  $\Omega$  is the symplectic metric  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  (such that  $\mathcal{S}^T \Omega \mathcal{S} = \Omega$ ) and  $\mathcal{V}_b$  is the bare  $\mathcal{V}$  (only bosonic part).

The  $(\psi, \lambda)$  Pauli term can be written in the form as in (2.21) and we identify in (2.26) the symplectic vector  $(Q, P)$  with  $\bar{\mathcal{U}}_{\dot{\alpha}}$ , and

$$\mathcal{T}_{\beta\gamma}^{\dot{\alpha}} = -\frac{1}{2} (\sigma^\mu)^{\dot{\alpha}}_{\beta} \psi_{\mu\gamma} .$$

The last Pauli term,  $e^{-1} \mathcal{L}_{\text{Pauli}}(\chi, \lambda)$ , has the form (2.21), with

$$H_{A\alpha\beta} = \frac{1}{2} \partial_i f_{AB} \lambda_\alpha^B \bar{\chi}_\beta^i .$$

This we rewrite in the form (2.26) using the following identifications (note that  $(\text{Im } f)_{AB}$  is the matrix of the kinetic terms of the vectors, and is thus invertible)

$$\begin{aligned} Q_{i\alpha}^A &\equiv (\text{Im } f)^{-1 AB} \partial_i f_{BC} \lambda_\alpha^C ; & P_{Ai\alpha} &\equiv \bar{f}_{AB} Q_{i\alpha}^B \\ \mathcal{T}_{\beta\gamma}^{i\alpha} &= \frac{i}{4} \delta_{(\beta}^{\alpha} \chi_{\gamma)}^i . \end{aligned}$$

To prove that these  $(Q, P)$  form a symplectic vector, one uses the following relations (which are in general true for  $f_{AB}$  replaced by  $\bar{\mathcal{N}}_{AB}$ ):

$$\begin{aligned} \tilde{f} &= (C + Df)(A + Bf)^{-1} = (A^T + fB^T)^{-1} (C^T + fD^T) \\ \partial_i \tilde{f} &= D \partial_i f (A + Bf)^{-1} - (C + Df)(A + Bf)^{-1} B \partial_i f (A + Bf)^{-1} \\ &= (A^T + fB^T)^{-1} \partial_i f (A + Bf)^{-1} . \\ \text{Im } \tilde{f} &= (A^T + fB^T)^{-1} (\text{Im } f) (A + B\bar{f})^{-1} \\ \tilde{\lambda} &= (A + Bf) \lambda \end{aligned} \quad (3.61)$$

These formulas then give automatically quartic fermionic terms as discussed in section 2.

We observe that the requirements for having symplectic transformations, (3.58) and (3.59), are in principle weaker than what is necessary to have an  $N = 2$  theory.

## 4 Duality symmetries

### 4.1 The facts

Duality transformations in generic  $N = 2$  supergravity theories are a different choice of the symplectic representative  $(X^\Lambda, F_\Lambda)$  of the underlying special geometry. If the fields  $\mathcal{F}_{\mu\nu}^{+\Lambda}, G_{\Lambda\mu\nu}^+$  have no electric or magnetic sources these dualities are simply a different equivalent choice of sections  $(X^\Lambda, F_\Lambda)$  since they are defined up to a symplectic transformation[3][18]. However if the gauge fields are coupled to (abelian) sources then duality transformations map theories into different theories with a duality transformed source. Since the matrix  $\mathcal{N}_{\Lambda\Sigma}$  plays the role of a coupling constant it is clear that in perturbation theory the only possible duality transformations are those with  $B = 0$  and have a lower triangular block form

$$\mathcal{S} = \begin{pmatrix} A & 0 \\ C & A^{T-1} \end{pmatrix} . \quad (4.1)$$

Under such change, the action changes in a total derivative which, up to fermion terms, is

$$\mathcal{L}'(A, C) = \mathcal{L} + \text{Im } \mathcal{F}^{-\Lambda} (C^T A)_{\Lambda\Sigma} \mathcal{F}^{-\Sigma} . \quad (4.2)$$

So the lagrangian is invariant up to a surface term. A duality transformation is a symmetry if

$$\tilde{\mathcal{N}}(\tilde{X}, \tilde{F}) = \mathcal{N}(\tilde{X}, \tilde{F}) . \quad (4.3)$$

If  $F_\Lambda = F_\Lambda(X)$  this implies

$$\tilde{F}(\tilde{X}) = F(\tilde{X}) . \quad (4.4)$$

Then using (3.27) we should have [3][38]

$$\begin{aligned} 2F[(A + B\mathcal{F})X] &= 2F + 2X^\Lambda (C^T B)_\Lambda{}^\Sigma F_\Sigma \\ &\quad + X^\Lambda (C^T A)_{\Lambda\Sigma} X^\Sigma + F_\Lambda (D^T B)^{\Lambda\Sigma} F_\Sigma , \end{aligned} \quad (4.5)$$

which is a functional relation for  $F$  given  $A, B, C, D$ . Note that because of (3.27) it may happen that  $\tilde{F}(\tilde{X}) = 0$ . This is so when  $\frac{\partial \tilde{X}^\Lambda}{\partial X^\Sigma}$  is not an invertible matrix.

## 4.2 Heterotic $N = 2$ superstring theories

In  $N = 2$  heterotic string theories, as the one obtained by the fermionic construction or by compactification on  $T_2 \times K_3$ , one often encounters classical moduli spaces which are locally of the form [39][40][19][41][10][42]

$$\frac{O(2, n_v)}{O(2) \times O(n_v)} \times \frac{O(4, n_h)}{O(4) \times O(n_h)} , \quad (4.6)$$

where  $n_v$  and  $n_h$  are respectively the number of the moduli in vector and hypermultiplets. If there are no charged massless hypermultiplets with respect to the gauge group  $U(1)^r$ , with  $r = n_v$ , we may avoid holomorphic anomalies [43–46] and the situation for this theory may be similar to the rigid Yang–Mills theory coupled to supergravity with an additional dilaton axion multiplet. According to the previous discussion, all perturbative duality symmetries are those for which the previous formula holds for a subgroup of lower triangular matrices

$$\begin{pmatrix} A & 0 \\ C & A^{T-1} \end{pmatrix} \quad (4.7)$$

with  $A^T C$  symmetric.

The  $(r+2) \times (r+2)$  block  $A$  contains the target space  $T$  duality and  $C$  contains the Peccei–Quinn axion symmetry [12] (for the definition of  $S$  in the  $N = 2$  context, see below)

$$S \rightarrow S + 1 . \quad (4.8)$$

These are the tree level stringy symmetries of the massive states with  $M = |Z|$  where  $Z$  is the central charge of the  $N = 2$  supersymmetry algebra. If the number of  $T$ -moduli is  $r$  then the duality symmetries are in  $Sp(2r+4; \mathbb{Z})$ .

An important point is that we would like to make the tree level (string) symmetry manifest. This means that the gauge fields

$$\mathcal{A}_\mu^A = (G_\mu, B_\mu, \mathcal{A}_\mu^A) \quad A = 2, \dots, r+1 \quad (4.9)$$

( $G_\mu$  is the graviphoton and the  $B_\mu$  is the vector of the dilaton–axion multiplet) should transform in the  $2+r$  dimensional (vector) representation of the target space duality symmetry

$$\mathcal{A}' = A\mathcal{A} ; \quad A^T \eta A = \eta ; \quad \eta_{\Lambda\Sigma} = \text{Diag}(1, 1, -1, -1, \dots) , \quad (4.10)$$



with  $A \in O(2, r; \mathbb{Z})$ . Under the axion Peccei–Quinn symmetry  $S \rightarrow S + 1$

$$\mathcal{A}^{\Lambda'} = \mathcal{A}^{\Lambda} \quad , \quad G_{\Lambda\mu\nu} \rightarrow G_{\Lambda\mu\nu} + \eta_{\Lambda\Sigma} \mathcal{F}_{\mu\nu}^{\Sigma} \quad , \quad (4.11)$$

where

$$\mathcal{N}_{\Lambda\Sigma}(S + 1) = \mathcal{N}_{\Lambda\Sigma}(S) + \eta_{\Lambda\Sigma} \quad . \quad (4.12)$$

This formulation is directly obtained by  $N = 2$  reduction of the standard form of the  $N = 4$  supergravity action[12][8] with a moduli space of the type  $O(6, r)/O(6) \times O(r)/\Gamma$  and duality group  $\Gamma = O(6, r; \mathbb{Z})$ . However to get this in a standard  $N = 2$  supergravity form, one must introduce  $2 + r$  symplectic sections  $(X^{\Lambda}, F_{\Lambda})$  ( $\Lambda = 0, 1, \dots, r + 1$ ) for which  $O(2, r)$  is block diagonal and the  $S \rightarrow S + 1$  shift is lower triangular. This formulation can be obtained by making a symplectic rotation, with  $\mathcal{S}$  given by

$$\mathcal{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} \quad , \quad (4.13)$$

from a representation in which only  $O(2) \times O(r)$  is block diagonal [47], namely

$$\begin{aligned} O(2, r) : \quad & \begin{pmatrix} A & 0 \\ 0 & \eta A \eta \end{pmatrix} = \mathcal{S} A_1 \mathcal{S}^{-1} \\ S \rightarrow S + 1 : \quad & \begin{pmatrix} \mathbb{1} & 0 \\ \eta & \mathbb{1} \end{pmatrix} = \mathcal{S} A_2 \mathcal{S}^{-1} \quad , \end{aligned} \quad (4.14)$$

where  $A_1, A_2$  are the matrices given in ref. [47]. The new sections are given explicitly by eqs. (3.28),(3.29),

$$\begin{aligned} \hat{X}^{\Lambda} &= \frac{1}{\sqrt{2}} (\delta_{\Lambda\Sigma} - F_{\Lambda\Sigma}) X^{\Sigma} \\ \hat{F}_{\Lambda} &= \frac{1}{\sqrt{2}} (\delta_{\Lambda\Sigma} + F_{\Lambda\Sigma}) X^{\Sigma} \quad , \end{aligned} \quad (4.15)$$

where the function

$$F = -\sqrt{X_i^2} \sqrt{X_{\alpha}^2} \quad i = 0, 1; \quad \alpha = 2, \dots, r + 1 \quad (4.16)$$

was obtained in ref. [47]. From (4.15),(4.16) one can verify that the  $\hat{X}^{\Lambda}, \hat{F}_{\Lambda}$  satisfy the constraints  $\hat{X}^{\Lambda} \eta_{\Lambda\Sigma} \hat{X}^{\Sigma} = \hat{F}_{\Lambda} \eta^{\Lambda\Sigma} \hat{F}_{\Sigma} = \hat{X}^{\Lambda} \hat{F}_{\Lambda} = 0$ . In particular, the new variables  $\hat{X}^{\Lambda}$  are not independent. The previous constraints imply that we may set

$$\hat{F}_{\Lambda} = S \eta_{\Lambda\Sigma} \hat{X}^{\Sigma} \quad (4.17)$$

and from eq. (3.27) we find  $\widehat{F}(\widehat{X}) = 0$ . Note that this is precisely the case for which  $\widehat{F}_\Lambda = \widehat{F}_\Lambda(\widehat{X}^\Lambda)$  does not hold.

Since  $O(2, r)$  is block diagonal, the new sections  $(\widehat{X}^\Lambda, \widehat{F}_\Lambda)$  are  $O(2, r)$  vectors. Recalling that the manifold  $\frac{O(2, r)}{O(2) \times O(r)}$  can be described by the following equations

$$\begin{aligned}\eta_{\Lambda\Sigma} \Phi^\Lambda \Phi^\Sigma &= 0 \\ \eta_{\Lambda\Sigma} \Phi^\Lambda \overline{\Phi}^\Sigma &= 1\end{aligned}\tag{4.18}$$

where  $\Phi^\Lambda$  are coordinates in  $CP(1, r)$ , we may actually set

$$\Phi^\Lambda = \frac{\widehat{X}^\Lambda}{\sqrt{\widehat{X}^\Sigma \eta_{\Sigma\Pi} \widehat{X}^\Pi}} .\tag{4.19}$$

The Kähler potential is

$$K = -\log i(\widehat{X}^\Lambda \widehat{F}_\Lambda - \widehat{\overline{X}}^\Lambda \widehat{F}_\Lambda) = -\log i(\overline{S} - S) - \log \widehat{X}^\Lambda \eta_{\Lambda\Sigma} \widehat{\overline{X}}^\Sigma .\tag{4.20}$$

Under  $S \rightarrow S + 1$

$$\begin{aligned}\widehat{X}^\Lambda &\rightarrow \widehat{X}^\Lambda \\ \widehat{F}_\Lambda &\rightarrow \widehat{F}_\Lambda + \eta_{\Lambda\Sigma} \widehat{X}^\Sigma .\end{aligned}\tag{4.21}$$

In the same basis the (non-perturbative) inversion  $S \rightarrow -\frac{1}{S}$  is given by the symplectic matrix  $\begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}$ . This element, together with the one corresponding to  $S \rightarrow S + 1$  generates an  $Sl(2, \mathbb{Z})$  commuting with the  $O(2r, \mathbb{Z})$  in  $Sp(2r + 4, \mathbb{Z})$ . The inversion is actually the only symmetry generator with  $B \neq 0$ . It leaves invariant (4.20) up to a Kähler transformation and it will be a symmetry of the classical spectrum (as it comes by truncation of the  $N = 4$  spectrum [12]) of electrically and magnetically charged states discussed in chapter 5 .

The holomorphic sections  $\widehat{X}^\Lambda$  can be written as [8]

$$\widehat{X}^\Lambda = \left( \frac{1}{2}(1 + y_\alpha^2), \frac{i}{2}(1 - y_\alpha^2), y^\alpha \right) ,\tag{4.22}$$

where the  $y^\alpha$  are coordinates of the  $O(2, r)/O(2) \times O(r)$  manifold. In terms of the  $\Phi$  variables the kinetic matrix  $\widehat{\mathcal{N}}_{\Lambda\Sigma}$  turns out to be [8][10][12]

$$\widehat{\mathcal{N}}_{\Lambda\Sigma}(\widehat{X}) = (S - \overline{S})(\Phi_\Lambda \overline{\Phi}_\Sigma + \overline{\Phi}_\Lambda \Phi_\Sigma) + \overline{S} \eta_{\Lambda\Sigma} ,\tag{4.23}$$

where  $\Phi_\Lambda = \eta_{\Lambda\Sigma} \Phi^\Sigma$ , and we will also further raise or lower indices with  $\eta$ .

Notice that (4.23) cannot be computed directly from (3.9) since in the new basis the denominator identically vanishes. On the other hand, one can use the formula (2.10), which in our case becomes

$$\hat{\mathcal{N}}(\hat{X}, \hat{F}) = (\mathbb{1} + \mathcal{N}(X))(\mathbb{1} - \mathcal{N}(X))^{-1} \quad (4.24)$$

and substitute for  $X^\Lambda$  the right hand side of the inverse transformations of (4.15)

$$\begin{aligned} X^\Lambda &= \frac{1}{\sqrt{2}}(\delta_{\Lambda\Sigma} + S\eta_{\Lambda\Sigma})\hat{X}^\Sigma \\ F_\Lambda &= \frac{1}{\sqrt{2}}(-\delta_{\Lambda\Sigma} + S\eta_{\Lambda\Sigma})\hat{X}^\Sigma . \end{aligned} \quad (4.25)$$

Formula (4.23) is precisely what is obtained from  $N = 4$  supergravity. Because of target space duality we expect that also the  $\hat{X}^\Lambda, \hat{F}_\Lambda$  become, because of one loop corrections, a lower triangular representation of  $Sp(2r + 4, \mathbb{Z})$

$$\begin{pmatrix} \hat{X}^\Lambda \\ \hat{F}_\Lambda \end{pmatrix} \rightarrow \begin{pmatrix} A & 0 \\ A^{T-1}C & A^{T-1} \end{pmatrix} \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} , \quad (4.26)$$

where the matrix  $C$  comes from the monodromy of the one-loop term [1,2].

It is interesting to compute explicitly the coupling of the dilaton to the vector fields. The vector kinetic term is

$$\text{Im } \overline{\mathcal{N}}_{\Lambda\Sigma} \mathcal{F}_{\mu\nu}^{-\Lambda} \mathcal{F}^{-\Sigma\mu\nu} = -2\text{Im } \mathcal{N}_{\Lambda\Sigma} \mathcal{F}_{\mu\nu}^\Lambda \mathcal{F}^{\Sigma\mu\nu} + \text{Re } \mathcal{N}_{\Lambda\Sigma} \mathcal{F}^{\Lambda\mu\nu} \tilde{\mathcal{F}}_{\mu\nu}^\Sigma . \quad (4.27)$$

and, in particular, setting in (4.22)  $y^\alpha = 0$ , it becomes

$$-2\text{Im } S(\mathcal{F}^0 \mathcal{F}^0 + \mathcal{F}^1 \mathcal{F}^1 + \mathcal{F}^\alpha \mathcal{F}^\alpha) + \text{Re } S(\mathcal{F}^0 \tilde{\mathcal{F}}^0 + \mathcal{F}^1 \tilde{\mathcal{F}}^1 - \mathcal{F}^\alpha \tilde{\mathcal{F}}^\alpha) . \quad (4.28)$$

We see that the dilaton couples in a universal way to the vectors while in the topological term we have a coupling with lorentzian signature.

### 4.3 Duality symmetries in $N > 2$ supergravities

The general considerations of section 2 about duality symmetries will apply to any higher  $N > 2$  extended supergravity theory. Therefore, it is worth to briefly mention the implications of duality symmetries for some non-perturbative properties that these theories may exhibit. The important fact about  $N > 2$  theories is that the scalar field space is (at least locally) a homogeneous symmetric space  $G/H$ , where  $G$  is some non compact subgroup of  $Sp(2n)$  ( $n$  is the total number of vector fields existing in the theory).  $H$  is its maximal compact subgroup, as it must be for the kinetic matrix of the scalar field space to be positive definite.

On general grounds, we also know that the fields  $(\mathcal{F}^{-A}, G_A^-)$  must belong to a linear representation of  $G$  which is given by the decomposition of the  $(2n)$ -dimensional vector representation of  $Sp(2n)$  under  $G$ . Thus, it is obvious that if this representation remains irreducible in  $G$ , the duality symmetry will necessarily mix electrically and magnetically charged states, since the  $Sp(2n)$  vector  $(n_{(m)}^A = 0, n_A^{(e)})$  cannot be an invariant vector of  $G$ .

It is now a fact of life that the full duality (continuous) symmetry  $G$  of any  $N > 2$  theory has a  $2n$  dimensional representation which remains irreducible under  $Sp(2n)$  (see table below [48]). This immediately implies that, if we assume, as conjectured in ref. [49], that the full  $G(\mathbb{Z})$  is a symmetry of the dyonic states, then  $G(\mathbb{Z})$  must be non-perturbative since the matrix  $B$  (see eq. (2.5)) in  $G(\mathbb{Z})$  will not be vanishing.  $N = 3, 5, 6$  supergravities can be obtained as low energy limits of  $d = 4$  string models [50].

Another implication of this conjecture, for the case of  $N = 4$  theories, is that, as pointed out in ref. [49], the spectrum of the BPS states of the ten dimensional heterotic string compactified on  $T_6$  should be identical to the spectrum of the same states for type II strings compactified on  $K_3 \times T_2$ , since the full  $N = 4$  BPS spectrum, invariant under  $Sl(2; \mathbb{Z}) \times SO(6, n-6; \mathbb{Z})$  is completely fixed by supersymmetry. This has the striking effect that at the non-perturbative level the type II theory should exhibit enhanced gauge symmetries equivalent to the  $N = 4$  heterotic string<sup>★</sup>.

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★ We acknowledge discussions with C. Hull on this point.

N	G	<i>repr.</i>
3	$SU(3, n-3)$	$(n_c)$
4	$SU(1, 1) \times SO(6, n-6)$	$(2, n)$
5	$SU(5, 1)$	$(20)$
6	$SO^*(12)$	$(32)$
8	$E_{7(7)}$	$(56)$

Table: Representations of  $G$  for  $(F^{-\Lambda}, G_{\Lambda}^{-})_{\Lambda=1,\dots,n}$  in extended supergravities

## 5 On monodromies in string effective field theories

### 5.1 Classical and quantum monodromies

We have just seen that the tree-level values of the symplectic sections  $(X^{\Lambda}(z), F_{\Lambda}(z))$  are given by

$$X^{\Lambda} \equiv X_{\text{tree}}^{\Lambda} \quad , \quad F_{\Lambda} = S \eta_{\Lambda\Sigma} X_{\text{tree}}^{\Sigma} \quad . \quad (5.1)$$

The target space duality group  $O(2, r; \mathbb{Z})$  acts non-trivially on them

$$\Gamma_{\text{cl}} : \quad \begin{pmatrix} X^{\Lambda} \\ F_{\Lambda} \end{pmatrix}_{\text{tree}} \rightarrow \begin{pmatrix} A & 0 \\ 0 & \eta A \eta \end{pmatrix} \begin{pmatrix} X^{\Lambda} \\ F_{\Lambda} \end{pmatrix}_{\text{tree}} \quad , \quad (5.2)$$

generalizing the action of the Weyl group of the rigid case [2].

At the one loop level, one expects that  $F_{\Lambda}^{\text{tree}}$  is changed to [46]

$$F_{\Lambda}^{\text{tree}} \rightarrow S X^{\Sigma} \eta_{\Lambda\Sigma} + f_{\Lambda}(X) \quad (5.3)$$

where  $f_{\Lambda}(X)$  is a modular covariant structure.

The associated perturbative monodromy can be obtained assuming, according to ref. [1], that the rigid perturbative monodromy does not affect the gravitational sector  $X^0, X^1, F_0, F_1$ . Thus the perturbative lower triangular monodromy matrix is  $\Gamma_{\text{cl}} T$ , where [1][2]

$$T = \begin{pmatrix} \mathbb{1} & 0 \\ C & \mathbb{1} \end{pmatrix} \quad (5.4)$$

and  $C$  is an  $(r+2) \times (r+2)$  symmetric matrix with non-vanishing entries on the  $r \times r$  block

$$C = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & & \\ \vdots & \vdots & & C_{ij} \\ 0 & 0 & & \end{pmatrix} \quad i, j = 1, \dots, r. \quad (5.5)$$

Indeed, we may think of decomposing  $Sp(4+2r)$  into  $Sp(4) \times Sp(2r)$  and simply assume that the rigid monodromy  $\Gamma_r \in Sp(2r)$  commute with the gravitational  $Sp(4)$  sector. This argument should at least apply when the vectors of the Cartan subalgebra of the enhanced gauge symmetry belong to the compact  $O(r)$  in  $O(2, r)$ .

In string theory, the classical stringy moduli space corresponds to the broken phase  $U(1)^r$  of several gauge groups with the same rank. For instance, for  $r = 2$ ,  $O(2, 2; \mathbb{Z})$  interpolates between  $SU(2) \times U(1)$ ,  $SU(2) \times SU(2)$  and  $SU(3)$ [51]. In the  $N = 4$  theory the  $O(6; 22)$  moduli space corresponds to broken phases of several gauge groups of rank 22 such as,  $U(1)^6 \times E_8 \times E_8$  or  $SO(32) \times U(1)^6$  or  $SO(44)$  which are not subgroups one of the other [39].

It is obvious that generically this means that the one loop  $\beta$ -function term [19][20] should have non-trivial monodromies at the points where some higher symmetry is restored. For instance, for  $r = 2$  we may expect non trivial monodromies around  $t = u$  ( $SU(2) \times U(1)$  symmetry restored) and  $t = u = i$ ,  $t = u = e^{2i\pi/3}$  ( $SU(2) \times SU(2)$  or  $SU(3)$  symmetry restored),  $t, u$  being the parameters defined below.

This means that in supergravity theories derived from strings, because of target space T-duality, the enhanced symmetry points are richer than in the rigid case. Since different enhancement points are consequence of  $O(2, r; \mathbb{Z})$  duality, we expect that a modular invariant treatment of quantum monodromies will automatically ensure non trivial monodromy at the enhanced symmetry points.

In the sequel we shall discuss in some more detail the classical and perturbative monodromies in the  $r = 1$  case ( $O(2, 1; \mathbb{Z})$ ) and the classical monodromies for  $r = 2$  ( $O(2, 2; \mathbb{Z})$ ).

Consider the tree level prepotential  $F$  in the so-called cubic form [3] for  $\frac{SU(1,1)}{U(1)} \times \frac{O(2,1)}{O(2)}$  :

$$F = \frac{1}{2}(X^0)^2 st^2, \quad (5.6)$$

where  $s = \frac{X^1}{X^0}$  is the dilaton coordinate and  $t = \frac{X^2}{X^0}$  is the single modulus of the classical target space duality. We parametrize the  $O(2, 1; \mathbb{Z})$  vector as follows

$$\begin{aligned} X^0 &= \frac{1}{2}(1 - t^2) \\ X^1 &= -t \\ X^2 &= -\frac{1}{2}(1 + t^2) \end{aligned} \quad (X^0)^2 + (X^1)^2 - (X^2)^2 = 0 \quad (5.7)$$

The symplectic transformation relating  $(X^\Lambda, F_\Lambda)$ ,  $(\Lambda = 0, 1, 2)$  to the  $(\hat{X}^\Lambda, \hat{F}_\Lambda)$  where  $O(2, 1)$  is linearly realized is easily found to be

$$\begin{pmatrix} \hat{X}^\Lambda \\ \hat{F}_\Lambda \end{pmatrix} = \begin{pmatrix} P & -2R \\ R & P' \end{pmatrix} \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}, \quad (5.8)$$

where

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}; \quad P' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}; \quad R = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}. \quad (5.9)$$

Let us now implement the  $t$ -modulus  $SL(2, \mathbb{Z})$  transformations  $t \rightarrow -\frac{1}{t}$ ,  $t \rightarrow t + n$  (note that while  $t \rightarrow -\frac{1}{t}$  corresponds to the  $SU(2)$  Weyl transformation of the rigid theory,  $t \rightarrow t + n$  has no counterpart in the rigid case, being of stringy nature). Using the parametrization (5.7) we find

$$\begin{aligned} t \rightarrow -\frac{1}{t} : \quad & \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv -\eta \in O(2, 1; \mathbb{Z}) \\ t \rightarrow t + n : \quad & \begin{pmatrix} 1 - \frac{n^2}{2} & n & \frac{n^2}{2} \\ -n & 1 & n \\ -\frac{n^2}{2} & n & 1 + \frac{n^2}{2} \end{pmatrix} \equiv V(n) \in O(2, 1; \mathbb{Z}). \end{aligned} \quad (5.10)$$

Note that (5.10) implies  $n \in 2\mathbb{Z}$ , i.e. the subgroup  $\Gamma_{(0)}(2)$  of  $SL(2, \mathbb{Z})$ . Actually this gives a projective representation in the subgroup in  $O(2, 1; \mathbb{Z})$  of the matrices congruent to the identity *mod* 2.

It follows that  $\Gamma_{\text{cl}}$  is generated by  $(\Gamma_1, \Gamma_2)$  where

$$\begin{aligned} \Gamma_1 &= \begin{pmatrix} -\eta & 0 \\ 0 & -\eta \end{pmatrix} \in Sp(6, \mathbb{Z}) \\ \Gamma_2 &= \begin{pmatrix} V(2) & 0 \\ 0 & \eta V(2)\eta \end{pmatrix} \in Sp(6, \mathbb{Z}). \end{aligned} \quad (5.11)$$

On the other hand it is possible to go to a stringy basis with a new metric  $X_0^2 + X_1^2 - X_2^2 = \tilde{X}_1^2 + 2XY$  such that  $SL(2, \mathbb{Z})$  is integral valued in  $O(2, 1; \mathbb{Z})$ .

The  $O(2, 1; \mathbb{Z})$  generators corresponding to translation and inversion are respectively given by:

$$\begin{pmatrix} 1 & -2n & 0 \\ 0 & 1 & 0 \\ n & -n^2 & 1 \end{pmatrix} ; \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} . \quad (5.12)$$

To make contact with the rigid theory it is convenient to define the inversion generator in  $O(2, 1; \mathbb{Z})$  with the opposite sign with respect to the previous definition.

Let us now examine the perturbative monodromy matrices  $T$ . If we assume as before that the  $t \rightarrow -\frac{1}{t}$  pertaining to the rigid theory does not affect the gravitational sector  $(X^0, X^1, F_0, F_1)$ , then we have

$$T = \begin{pmatrix} \eta & 0 \\ C & \eta \end{pmatrix} , \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad (5.13)$$

corresponding to the embedding of the  $Sp(2, \mathbb{Z})$  rigid transformations acting on the rigid section  $(X^2, F_2)$  in  $Sp(6, \mathbb{Z})$ . Furthermore, considering the transformation of the  $\mathcal{N}_{\Lambda\Sigma}$  matrix and setting  $D = A = \eta$ ,  $B = 0$  we find

$$\hat{\mathcal{N}}_{22} = -2 + \mathcal{N}_{22} \quad (5.14)$$

for all other entries  $\hat{\mathcal{N}}_{\Lambda\Sigma} = \mathcal{N}^{\Lambda\Sigma}$ . This is exactly the rigid result[1]. However conjugating the  $T$  matrix with  $\Gamma_2$  one gets

$$C_{\Lambda\Sigma} = \begin{pmatrix} 8 & -8 & -12 \\ -8 & 8 & 12 \\ -12 & 12 & 18 \end{pmatrix} \quad (5.15)$$

which shows that  $O(2, 1; \mathbb{Z})$  introduces non-trivial perturbative monodromies for all couplings. The other perturbative lower diagonal monodromy is the dilaton shift (4.14) which commutes with  $O(2, 1; \mathbb{Z})$ .



Analogous considerations hold for  $O(2, n; \mathbb{Z})$ ,  $n > 1$ . We limit ourselves to write down the generators of  $\Gamma_{\text{cl}}$  for the  $O(2, 2; \mathbb{Z})$  case. We use the parametrization of  $O(2, 2)/O(2) \times O(2)$  given by

$$\begin{aligned} X^0 &= \frac{1}{2}(1 - tu) \\ X^1 &= -\frac{1}{2}(t + u) \\ X^2 &= -\frac{1}{2}(1 + tu) \\ X^3 &= \frac{1}{2}(t - u) \quad (X^0)^2 + (X^1)^2 - (X^2)^2 - (X^3)^2 = 0, \end{aligned} \tag{5.16}$$

where  $t, u$  are the moduli appearing in the  $F$  function  $F = (X^0)^2 stu$ . In the same way as for the  $r = 1$  case it is easy to find the symplectic transformations relating the sections of the cubic parametrization to the  $X^\Lambda$  defined in (5.16). They are given by

$$\begin{pmatrix} X \\ F \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} X \\ F \end{pmatrix}, \tag{5.17}$$

with

$$\begin{aligned} X &= (X^0, X^1, X^2, X^3)^T, \quad F = (F_0, F_1, F_2, F_3)^T \\ A &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{5.18}$$

It is convenient to use the string basis where the metric  $\eta$  takes the form[12]

$$\eta = \begin{pmatrix} 0 & \mathbb{1}_{2 \times 2} \\ \mathbb{1}_{2 \times 2} & 0 \end{pmatrix}, \tag{5.19}$$

corresponding to the basis  $\frac{1}{\sqrt{2}}(X^0 \mp X^2), \frac{1}{\sqrt{2}}(X^1 \mp X^3)$ . Then one finds the following  $O(2, 2; \mathbb{Z})$  representation

$$\begin{aligned} ut &\rightarrow +\frac{1}{ut} : \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \gamma_{ut} \\ t &\rightarrow -\frac{1}{t} : \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} = \gamma_t \\ u &\rightarrow -\frac{1}{u} : \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} = \gamma_u \\ t &\rightarrow t + n : \begin{pmatrix} \mathbf{N}^t(-n) & 0 \\ 0 & \mathbf{N}(n) \end{pmatrix} = \gamma_n \\ t &\rightarrow u : \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \gamma; \quad a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{5.20}$$

where  $\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $\mathbf{N}(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ .

$\Gamma_{\text{cl}}$  is then generated by the matrices:

$$\Gamma_{ut} = \begin{pmatrix} \gamma_{ut} & 0 \\ 0 & \gamma_{ut} \end{pmatrix} ; \Gamma_t = \begin{pmatrix} \gamma_t & 0 \\ 0 & \gamma_t \end{pmatrix} ; \Gamma_u = \begin{pmatrix} \gamma_u & 0 \\ 0 & \gamma_u \end{pmatrix} ; \Gamma_n = \begin{pmatrix} \gamma_n & 0 \\ 0 & \gamma_n^T \end{pmatrix} . \quad (5.21)$$

We note that the points  $t = u$ ;  $t = u = i$ ;  $t = u = e^{\frac{2\pi i}{3}}$  are enhanced symmetry points corresponding to  $SU(2) \times U(1)$ ,  $SU(2) \times SU(2)$ , and  $SU(3)$  respectively [51]. Therefore we expect non-trivial quantum monodromies at these points according to the previous discussion.

## 5.2 The BPS mass formula

The classical and one loop monodromies are of course reflected in symmetries of the electrically charged massive states belonging to  $O(2, n; \mathbb{Z})$  lorentzian lattice[39]. The BPS mass formula [52] in the gravitational case is

$$M = |Z| = |n_{\Lambda}^{(e)} L^{\Lambda} - n_{(m)}^{\Lambda} M_{\Lambda}| = e^{K/2} |n_{\Lambda}^{(e)} X^{\Lambda} - n_{(m)}^{\Lambda} F_{\Lambda}| . \quad (5.22)$$

Note that the central charge  $Z$  has definite  $U(1)$  weight

$$Z \rightarrow e^{(\bar{f}-f)/2} Z , \quad (5.23)$$

while the mass  $M$  is Kähler invariant. The symplectic invariance of  $M$  also implies that  $(n_{(m)}^{\Lambda}, n_{\Lambda}^{(e)})$  transforms as  $(X^{\Lambda}, F_{\Lambda})$

$$\begin{pmatrix} n_{(m)}^{\Lambda} \\ n_{\Lambda}^{(e)} \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} n_{(m)}^{\Lambda} \\ n_{\Lambda}^{(e)} \end{pmatrix} , \quad (5.24)$$

where according to our previous discussion the perturbative symmetries have  $B = 0$ . Note that  $n_{(m)}^{\Lambda}, n_{\Lambda}^{(e)}$  must satisfy a lattice condition. In the tree level approximation we may write

$$M = |(n_{\Lambda}^{(e)} - n_{(m)}^{\Sigma} \eta_{\Lambda\Sigma} S) X^{\Lambda}| e^{K/2} \quad (5.25)$$

which is invariant under the tree level symmetry  $S \rightarrow S + 1$ , but also under the non-perturbative inversion  $S \rightarrow -\frac{1}{S}$ [34][13][12][14][15] taking into account that

$$K = -\log i(\bar{S} - S) - \log \frac{X^{\Lambda} \bar{X}^{\Sigma}}{M_{Pl}^2} \eta_{\Lambda\Sigma} . \quad (5.26)$$

Formula (5.25) is therefore invariant under the  $S - T$  duality symmetry  $Sl(2; \mathbb{Z}) \times O(2, r; \mathbb{Z}) \subset Sp(2r + 4; \mathbb{Z})$ .

The electric mass spectrum can be written as

$$M_{(e)}^2 = |Z|^2 = \frac{M_{Pl}^2}{2i(\bar{S} - S)} \mathcal{Q}^{\Lambda\Sigma} n_{\Lambda}^{(e)} n_{\Sigma}^{(e)} , \quad (5.27)$$

where  $i(\bar{S} - S) = \frac{8\pi}{g^2} > 0$  and  $\mathcal{Q}^{\Lambda\Sigma} = \Phi^{\Lambda} \bar{\Phi}^{\Sigma} + \bar{\Phi}^{\Lambda} \Phi^{\Sigma}$ . Formula (5.27) has exactly the same form as the analogous one obtained in  $N = 4$  (see ref [12]). When also magnetic charges are present, then

$$\begin{aligned} M^2 &= \frac{M_{Pl}^2}{i(\bar{S} - S)} (n_{\Lambda}^e - S n_{\Lambda}^m) \left( \frac{1}{2} \mathcal{Q}_{\Lambda\Sigma} - \frac{i}{2} \hat{\mathcal{Q}}_{\Lambda\Sigma} \right) (n_{\Sigma}^e - \bar{S} n_{\Sigma}^m) \\ &= \frac{M_{Pl}^2}{4} (n_m, n_e) (\mathcal{M} \mathcal{Q} + \mathcal{L} \hat{\mathcal{Q}}) \begin{pmatrix} n_m \\ n_e \end{pmatrix} , \end{aligned} \quad (5.28)$$

where  $\mathcal{M} = \frac{1}{\text{Im } S} \begin{pmatrix} S \bar{S} & -\text{Re } S \\ -\text{Re } S & 1 \end{pmatrix}$ ,  $\mathcal{L} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$  and  $\hat{\mathcal{Q}} = i(\Phi^{\Lambda} \bar{\Phi}^{\Sigma} - \bar{\Phi}^{\Lambda} \Phi^{\Sigma})$ . Recalling that  $\mathcal{Q}^{\Lambda\Sigma} = \frac{1}{2}(\eta^{\Lambda\Sigma} + \frac{\text{Im } \mathcal{N}^{\Lambda\Sigma}}{\text{Im } S})$ , this becomes

$$\begin{aligned} M^2 &= \frac{M_{Pl}^2}{i(\bar{S} - S)} (n_{\Lambda}^e - S n_{\Lambda}^m) \left[ \frac{1}{4} \left( \frac{\text{Im } \mathcal{N}_{\Lambda\Sigma}}{\text{Im } S} + \eta_{\Lambda\Sigma} \right) - \frac{i}{2} \hat{\mathcal{Q}}_{\Lambda\Sigma} \right] (n_{\Sigma}^e - \bar{S} n_{\Sigma}^m) \\ &= \frac{1}{4} M_{Pl}^2 (n_m, n_e) \left[ \frac{1}{2} \mathcal{M} \left( \frac{\text{Im } \mathcal{N}}{\text{Im } S} + \eta \right) + \mathcal{L} \hat{\mathcal{Q}} \right] \begin{pmatrix} n_m \\ n_e \end{pmatrix} . \end{aligned} \quad (5.29)$$

From this expression one can see that the antisymmetric term  $\hat{\mathcal{Q}}$  vanishes if

$$n_{\Lambda}^{(e)} = m_1 n_{\Lambda} \quad , \quad n_{(m)}^{\Lambda} = m_2 n_{\Sigma} \eta^{\Lambda\Sigma} , \quad (5.30)$$

or, as it happens for the perturbative string, if no magnetic states are present ( $n_{\Lambda}^m = 0$ ,  $n_{\Lambda}^e \equiv n_{\Lambda}$ ). In such case eq. (5.28) becomes

$$M^2 = \frac{M_{Pl}^2}{8 \text{Im } S} |m_1 - S m_2|^2 [n_{\Lambda} n_{\Sigma} (2 \mathcal{Q}^{\Lambda\Sigma} - \eta^{\Lambda\Sigma}) + n_{\Lambda} n_{\Sigma} \eta^{\Lambda\Sigma}] . \quad (5.31)$$

and since  $\text{Im } \mathcal{N}^{\Lambda\Sigma}$ , being the vector kinetic matrix, is always positive definite,

$$M^2 = 0 \iff n^{\Lambda} n^{\Sigma} \eta_{\Lambda\Sigma} < 0 \quad (n^{\Lambda} \neq 0) . \quad (5.32)$$

As an example, take  $O(2, 2; \mathbb{Z})$  and look for solutions of (5.32) corresponding to the string condition  $n^\Lambda n_\Lambda = -2$ . Using the parametrization (5.16) we have

$$\begin{aligned} n_\Lambda X^\Lambda &= n_0 X^0 + n_1 X^1 - n_2 X^2 - n_3 X^3 \\ &= \frac{1}{2}[(n_0 + n_2) - (n_1 + n_3)t - (n_1 - n_3)u - (n_0 - n_2)tu] \end{aligned} \quad (5.33)$$

Setting

$$\begin{aligned} n^0 + n^2 &= -p_2 \sqrt{2} \\ n^1 + n^3 &= q_1 \sqrt{2} \\ n^1 - n^3 &= -p_1 \sqrt{2} \\ n^0 - n^2 &= q_2 \sqrt{2} \\ n^\Lambda n_\Lambda &= (n^0 + n^2)(n^0 - n^2) + (n^1 + n^3)(n^1 - n^3) = -2(p_2 q_2 + p_1 q_1) = -2 \\ &\rightarrow p_2 q_2 + p_1 q_1 = 1 \end{aligned} \quad (5.34)$$

we have

$$n_\Lambda X^\Lambda = \frac{1}{\sqrt{2}}(-p_2 - q_1 t + p_1 u - q_2 t u) . \quad (5.35)$$

Let us verify that at the three enhancement points we get the correct number of massless states. If we take  $t = u$  ( $X^2 = 0$ ) we find

$$\begin{aligned} n_\Lambda X^\Lambda(t = u) &= \frac{1}{\sqrt{2}}[-p_2 - (q_1 - p_1)t - q_2 t^2] \\ &\rightarrow q_2 = p_2 = 0 \quad q_1 = p_1 = \pm 1 , \end{aligned}$$

yielding the two massless states  $(q_1, q_2) = (\pm 1, 0)$ . In particular, for  $t = u = i$  we have the solutions

$$\begin{aligned} n_\Lambda X^\Lambda(t = u = i) &= \frac{1}{\sqrt{2}}[-p_2 + q_2 - (q_1 - p_1)i] \\ &\rightarrow p_2 = q_2 \quad , \quad q_1 = p_1 \quad , \quad q_1^2 + q_2^2 = 1 \end{aligned} \quad (5.36)$$

yielding the four states  $(q_1, q_2) = (\pm 1, 0), (0, \pm 1)$ . Taking instead  $t = u = e^{2\pi i/3}$  (such that  $t^2 = \bar{t}$ ), we get

$$\begin{aligned} n_\Lambda X^\Lambda(t = u = e^{2\pi i/3}) &= 0 \\ &\rightarrow +\frac{1}{2}(q_1 + q_2 - p_1) - p_2 = 0 \quad , \quad q_1 - q_2 - p_1 = 0 \\ &\rightarrow p_1 = q_1 - q_2 \quad , \quad p_2 = q_2 \quad \rightarrow \quad q_1^2 + q_2^2 - q_1 q_2 = 1 \end{aligned} \quad (5.37)$$

yielding the six states  $(q_1, q_2) = (\pm 1, \pm 1), (\pm 1, 0), (0, \pm 1)$ . As expected, these massless states together with the two original  $(0, 0)$  states, fill the adjoint representation of  $SU(2) \otimes U(1)$  ( $t = u$ ),  $SU(2) \otimes SU(2)$  ( $t = u = i$ ),  $SU(3)$  ( $t = u = e^{2\pi i/3}$ ).

Unlike in  $N = 4$  theories, in  $N = 2$  theories the quantum spectrum will not coincide with the classical spectrum. It will be found by substituting  $F_{\Lambda\text{tree}} \equiv S\eta_{\Lambda\Sigma}X^\Sigma \rightarrow F_{\Lambda\text{tree}} + \text{quantum corrections}$  in (5.22).

## 6 Conclusions

In this paper we have formulated electromagnetic duality transformations in generic  $D = 4$ ,  $N = 2$  supergravities theories in a form suitable to investigate non-perturbative phenomena. Our formulation is manifestly duality covariant for the full Lagrangian, including fermionic terms, which unlike the rigid case, cannot be retrieved from the  $N = 1$  formulation, nor from the  $N = 2$  tensor calculus approach. Particular attention has been given to classical  $T$ -duality symmetries which actually occur in string compactifications and whose linear action on the gauge potential fields do not allow for the existence of a prepotential function  $F$  for the  $N = 2$  special geometry. As examples we described the “classical” electric and monopole spectrum for  $T$ -duality symmetries of the type  $O(2, r; \mathbb{Z})$ , with particular details for the  $r = 1, 2$  cases, by using the  $N = 2$  formalism.

For “classical” monodromies this spectrum is of course related to the spectrum of  $N = 4$  theories studied by Sen and Schwarz [12]. Possible extensions of duality symmetries to type II strings have been conjectured by Hull and Townsend [49] and also discussed in [2]. In the present context of  $N = 2$  heterotic strings the corresponding type II theories, having  $N = 2$  space-time supersymmetry would correspond to  $(2, 2)$  superconformal field theories, i.e. quantum Calabi–Yau manifolds.

Due to the non-compact symmetries the BPS saturated states with non-vanishing central charges have a spectrum quite different from the rigid case. Indeed in rigid theories the “classical” central charge  $Z_{(cl)}$  vanishes at the enhanced symmetry points where the original gauge group is restored since there is no dimensional scale other than the Higgs v.e.v.. On the contrary, in the supergravity theory the BPS spectrum at these particular points corresponds in general to electrically and magnetically charged states with Planckian mass (black holes, gravitational monopoles and dyons) [53, 12, 54–57]. The only charged states which become massless at the enhanced symmetric point are those with  $\eta^{\Lambda\Sigma}n_\Lambda^{(e)}n_\Sigma^{(e)} < 0$ .

We also discussed perturbative monodromies and their possible relations with the rigid case. Non perturbative duality symmetries are more difficult to guess, but it is tempting to conjecture that a quantum monodromy consistent with positivity of the metric and special geometry may be originated by a 3-dimensional Calabi-Yau manifold or its mirror image. If this is the case this manifold should embed in some sense the class of Riemann surfaces studied[1][2] in connection with the moduli space of  $N = 2$  rigid supersymmetric Yang-Mills theories.

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